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THE MOTION OF SUSPENDED PARTICLES ALMOST IN CONTACT

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Abstract—The forces and torques on two moving solid particles suspended in a fluid and almost in contact with each other (or on a particle almost in contact with a wall) are found in terms of their relative motion by using a type of lubrication theory, the results so obtained being asymptotically valid for small gap widths. It is assumed that the surfaces of the particles involved if brought together, are such that contact would occur at a *single* point at which surface curvatures are finite.

1. INTRODUCTION

Goldman, Cox & Brenner (1967), following the earlier work by O'Neill (1964) and by Dean & O'Neill (1963), investigated theoretically the slow motion of a solid sphere through a viscous fluid near a plane wall. Translation of the sphere parallel to the wall and rotation of the sphere about an axis parallel to the wall were considered. It was assumed that the fluid was at rest at infinity and that the fluid inertia was negligible in comparison to viscous effects so that the creeping motion equations were valid. These equations were solved by expressing them in terms of spherical bipolar coordinates. However the numerical computation of the "exact" solutions obtained, converged poorly if the gap width, a between sphere and plane became very much smaller than the sphere radius R. Thus in order to obtain exact solutions for such cases, Goldman, Cox & Brenner (1967) developed a lubrication theory which gave the force and torque on the sphere which was asymptotically correct for small values of the ratio a/R. Such asymptotic results agreed favourably with the "exact" solutions for the smallest values of a/R calculated. O'Neill & Stewartson (1967) obtained a more accurate solution asymptotically correct for small a/R by matching the lubrication theory valid within the gap onto an outer solution valid elsewhere obtained using tangent sphere coordinates.

The present paper extends the lubrication theory to obtain the forces and torques acting on any two solid surfaces separated by a viscous fluid, the surfaces being such that if they were to be brought together, contact would occur at a single point only. The forces and torques on the surfaces are found as the leading terms of asymptotic expansions valid for small gap widths. From such results the asymptotic forms of the resistance matrices for the surfaces are found. These results are then applied to the problem of calculating the forces and torques on two spheres (of unequal radii) almost in contact and on a sphere and cylinder almost in contact. In the former case, when the radius of one sphere tends to infinity the results reduce to those given by Goldman, Cox & Brenner (1967) whereas for the case in which sphere radii are equal, the results reduce to those given by Zia, Cox & Mason (1967). Also for the tangential motion of two unequal spheres, the results of O'Neill & Majumdar (1970) are obtained.

The implications and uses of this type of lubrication theory are discussed in the final section.

2. METHOD OF SOLUTION

Consider two particles suspended in a fluid of viscosity μ such that their surfaces W and W' are almost in contact and are such that if they are brought together, contact would occur at a single point at which the principal curvatures of both surfaces are finite. It is assumed that both surfaces are rigid and that their minimum separation distance, a, is very much smaller than the radii of curvature of the surfaces at the point of minimum separation. It is assumed also that both surfaces W and W' move in some prescribed manner and that it is required to find the forces and torques acting upon each of the surfaces as a result of this motion. The fluid in the gap between the surfaces is assumed to be incompressible and Newtonian and the Reynolds number for its motion so small that inertial effects may be neglected. The fluid velocity $\mathbf{u}$ and pressure p then satisfy the creeping motion equations

$$\mu\nabla^2\mathbf{u} - \nabla p = 0,$$

$$\nabla \cdot \mathbf{u} = 0, \qquad [2.1]$$

with given values of $\mathbf{u}$ on the surfaces W and W'. Rectangular Cartesian axes (x_1, x_2, x_3) are taken with origin O in the surface W at the point of minimum separation of the surfaces. The x_3 axis is taken to be normal to the surface W and hence also to the surface W'. The x_1 and x_2 axes are then chosen as tangents to the surface W in such a manner that they lie in the directions of the principal curvatures of the surface W at O. The surface W itself may then be written as

$$x_3 = -(x_1^2/2R_1) - (x_2^2/2R_2) + O(r^3), \qquad [2.2a]$$

for small values of $r = (x_1^2 + x_2^2)^{1/2}$, where R_1 and R_2 are the principal radii of curvature of the surface W at O. However if the surface W is symmetric in the sense that in [2.2a] the value of x_3 remains unchanged if x_1 and x_2 are replaced by $-x_1$ and $-x_2$ respectively, then the equation for the surface W must be of the form

$$x_3 = -(x_1^2/2R_1) - (x_2^2/2R_2) + O(r^4). \qquad [2.2b]$$

While the leading terms of the asymptotic forms of most of the resistance coefficients can be found for the general form of surface [2.2a], there are some which become singular for a surface of type [2.2a], even though they are non-singular for a surface of type [2.2b]. These latter values will not be calculated here since their evaluation requires knowledge of the $O(r^3)$ terms in [2.2a]. There are still other resistance coefficients which are never singular and their calculation will be shown to require knowledge of the complete particle shape and not just the form of the surface taken near the point O.

A new rectangular Cartesian coordinate system (x_1^*, x_2^*, x_3^*) is now chosen in such a manner that its origin O' lies both on the x_3-axis and the surface W'. The x_3^* axis is taken to coincide with the x_3 axis and the x_1^* and x_2^* axes chosen so as to lie in the directions of the principal curvatures of the surface W' (see figure 1). The surface W' may then be written as

$$x_3^* = (x_1^{*2}/2S_1) + (x_2^{*2}/2S_2) + O(r^{*3}), \qquad [2.3a]$$

for small values of $r^* = (x_1^{*2} + x_2^{*2})^{1/2}$, where S_1 and S_2 are the principal radii of curvature of the surface W' at O'. If this surface W' is symmetric so that replacing x_1^* and x_2^* by $-x_1^*$ and $-x_2^*$ leaves x_3^* unchanged, then the surface W' is of the form

$$x_3^* = (x_1^{*2}/2S_1) + (x_2^{*2}/2S_2) + O(r^{*4}). \qquad [2.3b]$$

Let ϕ be the angle between the x_1 and x_1^* axes (x_1 axis to the x_1^* axis in the positive sense). The x_1^*, x_2^*, x_3^* coordinates are then related to the x_1, x_2, x_3 coordinates by the relations

$$x_1^* = x_1 \cos\phi + x_2 \sin\phi, \qquad x_2^* = -x_1 \sin\phi + x_2 \cos\phi \qquad x_3^* = x_3 - a, \qquad [2.4]$$

so that the surface W' given by [2.3a] is rewritten in the form

$$x_3 = a + x_1^2\left(\frac{\cos^2\phi}{2S_1} + \frac{\sin^2\phi}{2S_2}\right) + x_1x_2\left(\frac{1}{S_1} - \frac{1}{S_2}\right)\sin\phi\cos\phi + x_2^2\left(\frac{\sin^2\phi}{2S_1} + \frac{\cos^2\phi}{2S_2}\right) + O(r^3). \qquad [2.5]$$

If $\mathbf{u} = (u_1, u_2, u_3)$ relative to the x_1, x_2, x_3 axis system then [2.1] is written as

$$\mu\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)u_1 - \frac{\partial p}{\partial x_1} = 0, \qquad \mu\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)u_2 - \frac{\partial p}{\partial x_2} = 0,$$
$$\mu\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)u_3 - \frac{\partial p}{\partial x_3} = 0, \qquad \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0. \qquad [2.6]$$

If $\mathbf{U}$ and $\mathbf{U}'$ are, respectively, the velocities of the walls W and W' at O and O' and if $\mathbf{\Omega}$ and $\mathbf{\Omega}'$ are respectively the angular velocities of W and W' then the no slip boundary

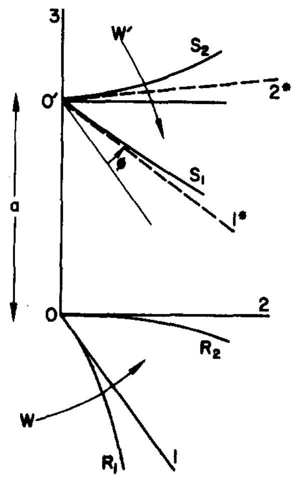

Figure 1. The surfaces W and W'.

condition for $\mathbf{u}$ at the wall W is

$$\mathbf{u} = \mathbf{U} + \mathbf{\Omega} \wedge \mathbf{r} \text{ on } W \qquad [2.7]$$

where

$$\mathbf{r} = [x_1, x_2, -(x_1^2/2R_1) - (x_2^2/2R_2) + O(r^3)], \qquad [2.8]$$

whilst the no slip boundary condition for $\mathbf{u}$ on the wall W' is

$$\mathbf{u} = \mathbf{U}' + \mathbf{\Omega}' \wedge \mathbf{r}' \text{ on } W' \qquad [2.9]$$

where

$$\mathbf{r}' = (x_1, x_2, x_3), \qquad [2.10]$$

x_3 being given by [2.5].

The velocity field $(\mathbf{u}, p)$ is expanded in terms of the gap separation a which is assumed to be very small. Such an expansion is singular, and it becomes necessary to discuss two regions of expansion. An outer region of expansion is defined using the coordinates (x_1, x_2, x_3) as independent variables and $\mathbf{u}$ and p as dependent variables. Thus in this region [2.6] with boundary conditions [2.7] and [2.9] are valid. However in the limit of $a \to 0$, the two surfaces touch, the point of contact being a singular point for the flow. Thus an inner region of expansion is defined using $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ as independent variables and $\tilde{\mathbf{u}}$ and $\tilde{p}$ as dependent variables where

$$\tilde{x}_1 = a^{-1/2}x_1, \tilde{x}_2 = a^{-1/2}x_2, \tilde{x}_3 = a^{-1}x_3 \qquad [2.11]$$

and

$$\tilde{u}_1 = a^{(1/2-k)}u_1, \tilde{u}_2 = a^{(1/2-k)}u_2, \tilde{u}_3 = a^{-k}u_3, \tilde{p} = a^{(2-k)}p, \qquad [2.12]$$

k being a constant as yet undefined.

By making use of [2.11] and [2.12], [2.6] may be converted to inner variables. It is then seen that $\tilde{\mathbf{u}}$ and $\tilde{p}$ must possess an expansion of the form

$$\tilde{\mathbf{u}} = \tilde{\mathbf{u}}_0 + a\mathbf{u}_1 + \cdots$$

$$\tilde{p} = \tilde{p}_0 + a\tilde{p}_1 + \cdots \qquad [2.13]$$

where the flow field $(\tilde{\mathbf{u}}_0, \tilde{p}_0)$ satisfies

$$\mu\frac{\partial^2(\tilde{u}_0)_1}{\partial\tilde{x}_3^2} - \frac{\partial\tilde{p}_0}{\partial\tilde{x}_1} = 0, \ \mu\frac{\partial^2(\tilde{u}_0)_2}{\partial\tilde{x}_3^2} - \frac{\partial\tilde{p}_0}{\partial\tilde{x}_2} = 0, \ \frac{\partial\tilde{p}_0}{\partial\tilde{x}_3} = 0, \ \frac{\partial(\tilde{u}_0)_1}{\partial\tilde{x}_1} + \frac{\partial(\tilde{u}_0)_2}{\partial\tilde{x}_2} + \frac{\partial(\tilde{u}_0)_3}{\partial\tilde{x}_3} = 0$$

$$[2.14]$$

and the flow field $(\tilde{\mathbf{u}}_1, \tilde{p}_1)$ satisfies

$$\mu\frac{\partial^2(\tilde{u}_1)_1}{\partial\tilde{x}_3^2} - \frac{\partial\tilde{p}_1}{\partial\tilde{x}_1} = -\mu\left(\frac{\partial^2}{\partial\tilde{x}_1^2} + \frac{\partial^2}{\partial\tilde{x}_2^2}\right)(\tilde{u}_0)_1, \ \mu\frac{\partial^2(\tilde{u}_1)_2}{\partial\tilde{x}_3^2} - \frac{\partial\tilde{p}_1}{\partial\tilde{x}_2} = -\mu\left(\frac{\partial^2}{\partial\tilde{x}_1^2} + \frac{\partial^2}{\partial\tilde{x}_2^2}\right)(\tilde{u}_0)_2$$

$$\frac{\partial\tilde{p}_1}{\partial\tilde{x}_3} = \mu\frac{\partial^2(\tilde{u}_0)_3}{\partial^2\tilde{x}_3^2}, \ \frac{\partial(\tilde{u}_1)_1}{\partial\tilde{x}_1} + \frac{\partial(\tilde{u}_1)_2}{\partial\tilde{x}_2} + \frac{\partial(\tilde{u}_1)_3}{\partial\tilde{x}_3} = 0. \qquad [2.15]$$

Relative to the inner variables [2.11], [2.2a] for the wall W takes the form

$$\tilde{x}_3 = -(\tilde{x}_1^2/2R_1) - (\tilde{x}_2^2/2R_2) + O(a^{1/2}) \qquad [2.16]$$

whilst [2.5] for the wall W' takes the form

$$\tilde{x}_3 = 1 + \tilde{x}_1^2\left(\frac{\cos^2\phi}{2S_1} + \frac{\sin^2\phi}{2S_2}\right) + \tilde{x}_1\tilde{x}_2\left(\frac{1}{S_1} - \frac{1}{S_2}\right)\sin\phi\cos\phi + \tilde{x}_2^2\left(\frac{\sin^2\phi}{2S_1} + \frac{\cos^2\phi}{2S_2}\right) + O(a^{1/2}). \qquad [2.17]$$

In order to solve [2.14] with boundary conditions on W and W', it is convenient to change variables to $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ where

$$\bar{x}_1 = \tilde{x}_1,\ \bar{x}_2 = \tilde{x}_2,\ \bar{x}_3 = \tilde{x}_3 + (\tilde{x}_1^2/2R_1) + (\tilde{x}_2^2/2R_2) \qquad [2.18]$$

so that the wall W becomes

$$\bar{x}_3 = O(a^{1/2}) \qquad [2.19]$$

and the wall W' becomes

$$\bar{x}_3 = h(\bar{x}_1, \bar{x}_2) + O(a^{1/2}) \qquad [2.20]$$

where the function $h(\bar{x}_1, \bar{x}_2)$ is defined as

$$h(\bar{x}_1, \bar{x}_2) = 1 + \bar{x}_1^2\left(\frac{1}{2R_1} + \frac{\cos^2\phi}{2S_1} + \frac{\sin^2\phi}{2S_2}\right) + \bar{x}_1\bar{x}_2\left(\frac{1}{S_1} - \frac{1}{S_2}\right)\sin\phi\cos\phi + \bar{x}_2^2\left(\frac{1}{2R_2} + \frac{\sin^2\phi}{2S_1} + \frac{\cos^2\phi}{2S_2}\right). \qquad [2.21]$$

With the new variables $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$, [2.14] for $(\tilde{\mathbf{u}}_0, \tilde{p}_0)$ take the form

$$\mu\frac{\partial^2(\tilde{u}_0)_1}{\partial\bar{x}_3^2} - \frac{\partial\tilde{p}_0}{\partial\bar{x}_1} - \frac{\bar{x}_1}{R_1}\frac{\partial\tilde{p}_0}{\partial\bar{x}_3} = 0,\ \mu\frac{\partial^2(\tilde{u}_0)_2}{\partial\bar{x}_3^2} - \frac{\partial p_0}{\partial\bar{x}_2} - \frac{\bar{x}_2}{R_2}\frac{\partial\tilde{p}_0}{\partial\bar{x}_3} = 0,\ \frac{\partial\tilde{p}_0}{\partial\bar{x}_3} = 0,$$

$$\frac{\partial(\tilde{u}_0)_1}{\partial\bar{x}_1} + \frac{\partial(\tilde{u}_0)_2}{\partial\bar{x}_2} + \frac{\partial(\tilde{u}_0)_3}{\partial\bar{x}_3} + \frac{\bar{x}_1}{R_1}\frac{\partial(\tilde{u}_0)_1}{\partial\bar{x}_3} + \frac{\bar{x}_2}{R_2}\frac{\partial(\tilde{u}_0)_2}{\partial\bar{x}_3} = 0. \qquad [2.22]$$

The boundary conditions [2.7] and [2.9] applicable on the walls W and W' may be expressed in inner variables and so one may write the boundary conditions for $\tilde{\mathbf{u}}_0$ in he form

$$(\tilde{u}_0)_1 = a^{-k}\{a^{1/2}U_1 - a\Omega_3\bar{x}_2 - a^{3/2}\Omega_2[(\bar{x}_1^2/2R_1) + (\bar{x}_2^2/2R_2)] + O(a^2)\},$$
$$(\tilde{u}_0)_2 = a^{-k}\{a^{1/2}U_2 + a\Omega_3\bar{x}_1 + a^{3/2}\Omega_1[(\bar{x}_1^2/2R_1) + (\bar{x}_2^2/2R_2)] + O(a^2)\},$$
$$(\tilde{u}_0)_3 = a^{-k}[U_3 + a^{1/2}(\Omega_1\bar{x}_2 - \Omega_2\bar{x}_1)] \text{ on } W \qquad [2.23]$$

and

$$(\tilde{u}_0)_1 = a^{-k}[a^{1/2}U_1' - a\Omega_3'\bar{x}_2 + a^{3/2}\Omega_2'h(\bar{x}_1, \bar{x}_2) + O(a^2)],$$
$$(\tilde{u}_0)_2 = a^{-k}[a^{1/2}U_2' + a\Omega_3'\bar{x}_1 - a^{3/2}\Omega_1'h(\bar{x}_1, \bar{x}_2) + O(a^2)],$$
$$(\tilde{u}_0)_3 = a^{-k}[U_3' + a^{1/2}(\Omega_1'\bar{x}_2 - \Omega_2'\bar{x}_1)] \text{ on } W'. \qquad [2.24]$$

Since the creeping motion equations are linear, and since $(\mathbf{u}, p)$ and hence $(\tilde{\mathbf{u}}_0, \tilde{p}_0)$ depend linearly upon $\mathbf{U}$, $\mathbf{U}'$, $\mathbf{\Omega}$ and $\mathbf{\Omega}'$, it follows that

$$\tilde{\mathbf{u}}_0 = \tilde{\mathbf{u}}_A + \tilde{\mathbf{u}}_B + \tilde{\mathbf{u}}_C, \qquad \tilde{p}_0 = \tilde{p}_A + \tilde{p}_B + \tilde{p}_C \tag{2.25}$$

where the flow fields $(\tilde{\mathbf{u}}_A, \tilde{p}_A)$, $(\tilde{\mathbf{u}}_B, \tilde{p}_B)$ and $(\tilde{\mathbf{u}}_C, \tilde{p}_C)$ each individually satisfy [2.14] and where $(\tilde{\mathbf{u}}_A, \tilde{p}_A)$ is the flow resulting from U_3 and U'_3, $(\tilde{\mathbf{u}}_B, \tilde{p}_B)$ is the flow resulting from U_1, U_2, Ω_1, Ω_2, U'_1, U'_2, Ω'_1 and Ω'_2 and $(\tilde{\mathbf{u}}_C, \tilde{p}_C)$ is the flow resulting from Ω_3 and Ω'_3. Thus $(\tilde{\mathbf{u}}_A, \tilde{p}_A)$ satisfies the boundary conditions

$$(\tilde{u}_A)_1 = (\tilde{u}_A)_2 = 0, \qquad (\tilde{u}_A)_3 = a^{-k}U_3 \qquad \text{on } W \tag{2.26}$$

$$(\tilde{u}_A)_1 = (\tilde{u}_A)_2 = 0, \qquad (\tilde{u}_A)_3 = a^{-k}U'_3 \qquad \text{on } W' \tag{2.27}$$

whilst $(\tilde{\mathbf{u}}_B, \tilde{p}_B)$ satisfies $(\tilde{u}_B)_1 = a^{-k+1/2}U_1 + O(a^{-k+3/2})$

$$(\tilde{u}_B)_2 = a^{-k+1/2}U_2 + O(a^{-k+3/2}), \qquad (\tilde{u}_B)_3 = a^{-k+1/2}(\Omega_1\bar{x}_2 - \Omega_2\bar{x}_1) \qquad \text{on } W \tag{2.28}$$

and $(\tilde{u}_B)_1 = a^{-k+1/2}U'_1 + O(a^{-k+3/2})$

$$(\tilde{u}_B)_2 = a^{-k+1/2}U'_2 + O(a^{-k+3/2}), \qquad (\tilde{u}_B)_3 = a^{-k+1/2}(\Omega'_1\bar{x}_2 - \Omega'_2\bar{x}_1) \qquad \text{on } W' \tag{2.29}$$

and $(\tilde{\mathbf{u}}_C, \tilde{p}_C)$ satisfies

$$(\tilde{u}_C)_1 = a^{-k+1}(-\Omega_3\bar{x}_2), \qquad (\tilde{u}_C)_2 = a^{-k+1}(\Omega_3\bar{x}_1), \qquad (\tilde{u}_C)_3 = 0 \qquad \text{on } W \tag{2.30}$$

and

$$(\tilde{u}_C)_1 = a^{-k+1}(-\Omega'_3\bar{x}_2), \qquad (\tilde{u}_C)_2 = a^{-k+1}(\Omega'_3\bar{x}_1), \qquad (\tilde{u}_C)_3 = 0 \qquad \text{on } W'. \tag{2.31}$$

The walls W and W' are taken to be

$$\bar{x}_3 = 0 \tag{2.32}$$

and

$$\bar{x}_3 = h(x_1, x_2) \tag{2.33}$$

with an error $O(a^{+1/2})$ in these expressions (see [2.19] and [2.20]).

From the boundary conditions [2.26] and [2.31] it is seen that in order to calculate $(\tilde{\mathbf{u}}_A, \tilde{p}_A)$ one must take $k = 0$, whilst for $(\tilde{\mathbf{u}}_B, \tilde{p}_B)$, $k = 1/2$ and for $(\tilde{\mathbf{u}}_C, \tilde{p}_C)$, $k = 1$.

It should be noted that the inner flow fields when expressed in terms of outer variables by [2.11] and [2.12] must be matched onto the outer expansion. It will be shown later that it is unnecessary to calculate in detail this outer expansion in order to find the force and torque exerted on W and W' to the order in a with which we shall be concerned. However, it should be noted that the outer expansion cannot contain any terms which tend to infinity as $a \to 0$ for a fixed value of $\mathbf{r}$, although any term in this expression may tend to infinity as $\mathbf{r}$ tends to the contact point.

3. DIRECT APPROACH OF SURFACES

From the boundary conditions [2.26] and [2.27] the flow field $(\tilde{\mathbf{u}}_A, \tilde{p}_A)$ of the inner expansion is that which results from the direct approach of the two surfaces (i.e. it results from U_3 and U'_3). Thus taking $k = 0$, the boundary conditions on $\tilde{\mathbf{u}}_A$ become

$$(\tilde{u}_A)_1 = (\tilde{u}_A)_2 = 0, \qquad (\tilde{u}_A)_3 = U_3 \qquad \text{on } \bar{x}_3 = 0, \tag{3.1}$$

$$(\tilde{u}_A)_1 = (\tilde{u}_A)_2 = 0, \quad (\tilde{u}_A)_3 = U'_3 \quad \text{on } \bar{x}_3 = h(\bar{x}_1, \bar{x}_2) \qquad [3.2]$$

in the limit of $a \to 0$.

Since $(\tilde{\mathbf{u}}_A, \tilde{p}_A)$ satisfies [2.22], it follows that

$$\tilde{p}_A = \tilde{p}_A(\bar{x}_1, \bar{x}_2), \quad (\tilde{u}_A)_1 = \frac{1}{2\mu}\frac{\partial \tilde{p}_A}{\partial \bar{x}_1}\bar{x}_3^2 + A\bar{x}_3 + C, \quad (\tilde{u}_A)_2 = \frac{1}{2\mu}\frac{\partial \tilde{p}_A}{\partial \bar{x}_2}\bar{x}_3^2 + B\bar{x}_3 + D \qquad [3.3]$$

where A, B, C and D are constants obtained from the $\bar{x}_3$ integration and are thus arbitrary functions of $\bar{x}_1$ and $\bar{x}_2$. These quantities A, B, C and D may be determined from the boundary conditions [3.1] and [3.2] as

$$A = -\frac{1}{2\mu}\left(\frac{\partial \tilde{p}_A}{\partial \bar{x}_1}\right) h(\bar{x}_1, \bar{x}_2) \quad B = -\frac{1}{2\mu}\left(\frac{\partial \tilde{p}_A}{\partial \bar{x}_2}\right) h(\bar{x}_1, \bar{x}_2) \quad C = D = 0. \qquad [3.4]$$

Substituting the expressions for $(\tilde{u}_A)_1$ and $(\tilde{u}_A)_2$ given by [3.3] into the last of [2.22] and integrating with respect to $\bar{x}_3$ one obtains

$$(\tilde{u}_A)_3 = -\frac{1}{6\mu}\left(\frac{\partial^2 \tilde{p}_A}{\partial \bar{x}_1^2} + \frac{\partial^2 \tilde{p}_A}{\partial \bar{x}_2^2}\right)\bar{x}_3^3 - \frac{1}{2}\left(\frac{\partial A}{\partial \bar{x}_1} + \frac{\partial B}{\partial \bar{x}_2}\right)\bar{x}_3^2$$

$$-\frac{1}{2\mu}\left(\frac{\bar{x}_1}{R_1}\frac{\partial \tilde{p}_A}{\partial \bar{x}_1} + \frac{\bar{x}_2}{R_2}\frac{\partial \tilde{p}_A}{\partial \bar{x}_2}\right)\bar{x}_3^2 - \left(\frac{A}{R_1}\bar{x}_1 + \frac{B}{R_2}\bar{x}_2\right)\bar{x}_3 + E \qquad [3.5]$$

where E is an arbitrary function of $\bar{x}_1$ and $\bar{x}_2$.

Since $(\tilde{u}_A)_3 = U_3$ on $\bar{x}_3 = 0$, it follows that

$$E = U_3. \qquad [3.6]$$

Also since $(\tilde{u}_A)_3 = U'_3$ on $\bar{x}_3 = h(\bar{x}_1, \bar{x}_2)$,

$$(U'_3 - U_3) = -\frac{1}{6\mu}\left(\frac{\partial^2 \tilde{p}_A}{\partial \bar{x}_1^2} + \frac{\partial^2 p_A}{\partial \bar{x}_2^2}\right)h^3 - \frac{1}{2}\left(\frac{\partial A}{\partial \bar{x}_1} + \frac{\partial B}{\partial \bar{x}_2}\right)h^2$$

$$-\frac{1}{2\mu}\left(\frac{\bar{x}_1}{R_1}\frac{\partial \tilde{p}_A}{\partial \bar{x}_1} + \frac{\bar{x}_2}{R_2}\frac{\partial \tilde{p}_A}{\partial \bar{x}_2}\right)h^2 - \left(\frac{A}{R_1}\bar{x}_1 + \frac{B}{R_2}\bar{x}_2\right)h. \qquad [3.7]$$

Substitution of the values of A and B from [3.4] into [3.7] yields

$$\frac{1}{12\mu}h^3\bar{\nabla}^2\tilde{p}_A + \frac{1}{4\mu}h^2(\bar{\nabla}\tilde{p}_A \cdot \bar{\nabla}h) = (U'_3 - U_3) \qquad [3.8]$$

in which $\tilde{p}_A$ and h are functions of $\bar{x}_1$ and $\bar{x}_2$ and in which all derivatives are taken with respect to the variables $\bar{x}_1$ and $\bar{x}_2$. Equation [3.8] may be written as

$$\bar{\nabla}\cdot(h^3\bar{\nabla}\tilde{p}_A) = 12\mu(U'_3 - U_3) \qquad [3.9]$$

where $h(\bar{x}_1, \bar{x}_2)$ is defined by [2.21]. One may write this expression for h in the form

$$h = 1 + \bar{\mathbf{x}}^T \cdot \mathbf{K} \cdot \bar{\mathbf{x}} \qquad [3.10]$$

where $\bar{\mathbf{x}}$ is the column vector

$$\bar{\mathbf{x}} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} \qquad [3.11]$$

and $\bar{\mathbf{x}}^T$ is its transpose. Then $\mathbf{K}$ is the matrix

$$\mathbf{K} = \begin{pmatrix} \dfrac{1}{2R_1} + \dfrac{\cos^2\phi}{2S_2} + \dfrac{\sin^2\phi}{2S_2} & \dfrac{\sin\phi\cos\phi}{2}\left(\dfrac{1}{S_1} - \dfrac{1}{S_2}\right) \\ \dfrac{\sin\phi\cos\phi}{2}\left(\dfrac{1}{S_1} - \dfrac{1}{S_2}\right) & \dfrac{1}{2R_2} + \dfrac{\sin^2\phi}{2S_1} + \dfrac{\cos^2\phi}{2S_2} \end{pmatrix}. \qquad [3.12]$$

Define λ_1 and λ_2 as the eigenvalues and $\bar{\mathbf{x}}_1^*$ and $\bar{\mathbf{x}}_2^*$ as the corresponding normalized eigenfunctions of the matrix $\mathbf{K}$. Thus λ_1 and λ_2 are the roots of

$$|\mathbf{K} - \lambda\mathbf{I}| = 0 \qquad [3.13]$$

where $\mathbf{I}$ is the identity matrix and $\bar{\mathbf{x}}_1^*$ and $\bar{\mathbf{x}}_2^*$ satisfy

$$\mathbf{K}\cdot\bar{\mathbf{x}}_1^* = \lambda_1\bar{\mathbf{x}}_1^* \qquad \mathbf{K}\cdot\bar{\mathbf{x}}_2^* = \lambda_2\bar{\mathbf{x}}_2^* \qquad [3.14]$$

and

$$\bar{\mathbf{x}}_1^{*T}\cdot\bar{\mathbf{x}}_1^* = \bar{\mathbf{x}}_2^{*T}\cdot\bar{\mathbf{x}}_2^* = 1. \qquad [3.15]$$

If a matrix $\mathbf{A}$ is defined as

$$\mathbf{A} = (\bar{\mathbf{x}}_1^*, \bar{\mathbf{x}}_2^*), \qquad [3.16]$$

then the transformation

$$\bar{\mathbf{x}} = \mathbf{A}\cdot\hat{\mathbf{x}}, \qquad [3.17]$$

where

$$\hat{\mathbf{x}} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix}, \qquad [3.18]$$

is orthogonal and transforms [3.10] for $h(\bar{x}_1, \bar{x}_2)$ into the form

$$h = 1 + \lambda_1\hat{x}_1^2 + \lambda_2\hat{x}_2^2. \qquad [3.19]$$

Thus [3.9] for the pressure $\tilde{p}_A$ expressed in terms of the $\hat{x}_1, \hat{x}_2$ variables becomes

$$\frac{\partial}{\partial\hat{x}_1}\left[(1 + \lambda_1\hat{x}_1^2 + \lambda_2\hat{x}_2^2)\frac{\partial\tilde{p}_A}{\partial\hat{x}_1}\right] + \frac{\partial}{\partial\hat{x}_2}\left[(1 + \lambda_1\hat{x}_1^2 + \lambda_2\hat{x}_2^2)\frac{\partial\tilde{p}_A}{\partial\hat{x}_2}\right] = 12\mu(U_3' - U_3). \qquad [3.20]$$

In order to solve this equation, the substitution

$$\hat{x}_1 = (1/\lambda_1^{1/2})\hat{r}\cos\theta \qquad \hat{x}_2 = (1/\lambda_2^{1/2})\hat{r}\sin\theta \qquad [3.21]$$

is made so that the equation takes the form

$$(\lambda_1\cos^2\theta + \lambda_2\sin^2\theta)\hat{r}^2(\partial^2\tilde{p}_A/\partial\hat{r}^2) + 2(\lambda_2 - \lambda_1)\sin\theta\cos\theta\hat{r}(\partial^2\tilde{p}_A/\partial\hat{r}\partial\theta)$$
$$+ (\lambda_1\sin^2\theta + \lambda_2\cos^2\theta)(\partial^2\tilde{p}_A/\partial\theta^2) + (\lambda_1\sin^2\theta + \lambda_2\cos^2\theta)\hat{r}(\partial\tilde{p}_A/\partial\hat{r})$$

$$+ 2(\lambda_1 - \lambda_2)\sin\theta\cos\theta(\partial\tilde{p}_A/\partial\theta) + (6\hat{r}^3/(1 + \hat{r}^2))(\lambda_1\cos^2\theta + \lambda_2\sin^2\theta)(\partial\tilde{p}_A/\partial\hat{r})$$
$$+ (6\hat{r}^2/(1 + \hat{r}^2))(\lambda_2 - \lambda_1)\sin\theta\cos\theta(\partial\tilde{p}_A/\partial\theta) = [12\mu(U_3' - U_3)\hat{r}^2/(1 + \hat{r}^2)^3]. \quad [3.22]$$

If it is assumed that $\tilde{p}_A$ is of order $\tilde{r}^n$ as $\tilde{r} \to \infty$, then by [3.3], [3.4] and [3.5] it is seen that $(\tilde{u}_A)_1$ and $(\tilde{u}_A)_2$ are $O(\tilde{r}^{n-1})$ as $\tilde{r} \to \infty$ and $(\tilde{u}_A)_3$ is of the form $U_3 + O(\tilde{r}^n)$ as $\tilde{r} \to \infty$. By expressing these quantities in outer variables and noting that the pressure and velocity in the outer region of expansion cannot contain any terms which tend to infinity as $a \to 0$, it may readily be shown that $n \leq -4$. Hence

$$\tilde{p}_A = O(\hat{r}^{-4}) \qquad \text{as } \hat{r} \to \infty. \quad [3.23]$$

Now the solution of [3.22] which satisfies [3.23] and possesses no singularity at $\tilde{r} = 0$ may be shown to be

$$\tilde{p}_A = -[3\mu(U_3' - U_3)/(\lambda_1 + \lambda_2)][1/(1 + \hat{r}^2)^2] + O(a^{1/2}). \quad [3.24]$$

The error term of order $a^{1/2}$ in the expression for $\tilde{p}_A$ arises from the fact that the expressions for the walls given in the boundary conditions [3.1] and [3.2] have an error of order $a^{1/2}$. If $\mathbf{n}$ is defined as the unit normal to the surfaces W from solid to fluid, then the force $\mathbf{F}$ and torque $\mathbf{G}$ about O exerted by the fluid on W are given by

$$F_i = \int_W p_{ij} n_j \, dS \quad [3.25]$$

and

$$G_i = \int_W \varepsilon_{ijk} r_j p_{kl} n_l \, dS \quad [3.26]$$

where p_{ij} is the stress tensor corresponding to $(\mathbf{u}, p)$ and dS is an element of area of the surface. It may readily be shown that for small values of $r = (x_1^2 + x_2^2)^{1/2}$, $\mathbf{n}$, dS and $\mathbf{r}$ are

$$\mathbf{n} = (x_1/R_1 + O(r^2), x_2/R_2 + O(r^2), 1 + O(r^2)), \qquad dS = dx_1\, dx_2(1 + O(r^2))$$
$$\mathbf{r} = (x_1, x_2, -x_1^2/2R_1 - x_2^2/2R_2 + O(r^3)). \quad [3.27]$$

For the evaluation of the integrals [3.25] and [3.26] the surface of W is divided into two ares S_ε and Σ_ε, the area S_ε being defined by that part of W for which the vector

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad [3.28]$$

satisfies the relation

$$\mathbf{x}^T \cdot \mathbf{K} \cdot \mathbf{x} \leq \varepsilon^2, \quad [3.29]$$

where $\mathbf{K}$ is the matrix given by [3.12] and ε is a small positive arbitrary parameter independent of a. The area Σ_ε is that part of the surface of W for which [3.29] is not satisfied (see figure 2).

The condition [3.29] may by [2.11] and [2.18] be written in terms of inner variables as

$$\bar{\mathbf{x}}^T \cdot \mathbf{K} \cdot \bar{\mathbf{x}} \leq a^{-1}\varepsilon^2 \quad [3.30]$$

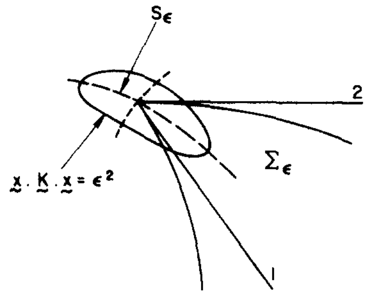

Figure 2. The division of W into areas S_ε and Σ_ε.

and by the comparison of [3.10] and [3.19] be further reduced to

$$\lambda_1\hat{x}_1^2 + \lambda_2\hat{x}_2^2 \le a^{-1}\varepsilon^2. \qquad [3.31]$$

Writing this condition in terms of $\hat{r}$ by [3.21], one obtains

$$\hat{r} \le a^{-1/2}\varepsilon. \qquad [3.32]$$

Thus the force $\mathbf{F}$ and torque $\mathbf{G}$ given by [3.25] and [3.26] may be written as

$$F_i = \int_{S_\varepsilon} p_{ij}n_j\,\mathrm{d}S + \int_{\Sigma_\varepsilon} p_{ij}n_j\,\mathrm{d}S \qquad [3.33]$$

$$G_i = \int_{S_\varepsilon} \varepsilon_{ijk}r_jp_{kl}n_l\,\mathrm{d}S + \int_{\Sigma_\varepsilon} \varepsilon_{ijk}r_jp_{kl}n_l\,\mathrm{d}S. \qquad [3.34]$$

Since ε is independent of a, the integrals over Σ_ε in [3.33] and [3.34] tend to finite limits as $a \to 0$, although each is a function of ε and may be singular as $\varepsilon \to 0$.

The integrals over S_ε are evaluated by expressing all quantities in inner variables, the region of integration then being defined by [3.32]. The values of $\mathbf{n}$, $\mathrm{d}S$ and $\mathbf{r}$ given by [3.27], when expressed in inner variables become

$$\mathbf{n} = \{a^{1/2}(\tilde{x}_1/R_1) + O(a), a^{1/2}(\tilde{x}_2/R_2) + O(a), 1 + O(a)\}, \qquad \mathrm{d}S = a\,\mathrm{d}\tilde{x}_1\,\mathrm{d}\tilde{x}_2\{1 + O(a)\}$$

$$\mathbf{r} = \{a^{1/2}\tilde{x}_1, a^{1/2}\tilde{x}_2, -a((\tilde{x}_1^2/2R_1) + (\tilde{x}_1^2/2R_2)) + O(a^{3/2})\}. \qquad [3.35]$$

The stress tensor p_{ij} is written in terms of inner variables by [2.11] and [2.12] as

$$p_{11} = a^k[-a^{-2}\tilde{p} + 2a^{-1}\mu(\partial\tilde{u}_1/\partial\tilde{x}_1)], \qquad p_{22} = a^k[-a^{-2}\tilde{p} + 2a^{-1}\mu(\partial\tilde{u}_2/\partial\tilde{x}_2)]$$

$$p_{33} = a^k[-a^{-2}\tilde{p} + 2a^{-1}\mu(\partial\tilde{u}_3/\partial\tilde{x}_3)]$$

$$p_{12} = p_{21} = a^k[a^{-1}\mu(\partial\tilde{u}_1/\partial\tilde{x}_2) + a^{-1}\mu(\partial\tilde{u}_2/\partial\tilde{x}_1)]$$

$$p_{23} = p_{32} = a^k[a^{-3/2}\mu(\partial\tilde{u}_2/\partial\tilde{x}_3) + a^{-1/2}\mu(\partial\tilde{u}_3/\partial\tilde{x}_2)]$$

$$p_{31} = p_{13} = a^k[a^{-3/2}\mu(\partial\tilde{u}_1/\partial\tilde{x}_3) + a^{-1/2}\mu(\partial\tilde{u}_3/\partial\tilde{x}_1)]. \qquad [3.36]$$

Substituting the values of $\mathbf{n}$, $\mathrm{d}S$, $\mathbf{r}$ and p_{ij} given by [3.35] and [3.36] into the integrals over

S_ε in [3.33] and [3.34], one obtains

$$\int_{S_\varepsilon} p_{1j}n_j \, dS = a^{k-1/2} \int_{S_\varepsilon} [-(\tilde{x}_1\tilde{p}/R_1) + \mu(\partial\tilde{u}_1/\partial\tilde{x}_3)] \, d\tilde{x}_1 \, d\tilde{x}_2 + O(a^k)$$

$$\int_{S_\varepsilon} p_{2j}n_j \, dS = a^{k-1/2} \int_{S_\varepsilon} [-(\tilde{x}_2\tilde{p}/R_2) + \mu(\partial\tilde{u}_2/\partial\tilde{x}_3)] \, d\tilde{x}_1 \, d\tilde{x}_2 + O(a^k)$$

$$\int_{S_\varepsilon} p_{3j}n_j \, dS = a^{k-1} \int_{S_\varepsilon} (-\tilde{p}) \, dx_1 \, d\tilde{x}_2 + O(a^k) \qquad [3.37]$$

and

$$\int_{S_\varepsilon} \varepsilon_{1jk}r_j p_{kl}n_l \, dS = a^{k-1/2} \int_{S_\varepsilon} (-\tilde{x}_2\tilde{p}) \, d\tilde{x}_1 \, d\tilde{x}_2 + O(a^{k+1/2})$$

$$\int_{S_\varepsilon} \varepsilon_{2jk}r_j p_{kl}n_l \, dS = a^{k-1/2} \int_{S_\varepsilon} (+\tilde{x}_1\tilde{p}) \, d\tilde{x}_1 \, d\tilde{x}_2 + O(a^{k+1/2})$$

$$\int_{S_\varepsilon} \varepsilon_{3jk}r_j p_{kl}n_l \, dS = a^k \int_{S_\varepsilon} \{[(1/R_1) - (1/R_2)]\tilde{x}_1\tilde{x}_2\tilde{p} + \mu[\tilde{x}_1(\partial\tilde{u}_2/\partial\tilde{x}_3) - \tilde{x}_2(\partial\tilde{u}_1/\partial\tilde{x}_3)]\} \, d\tilde{x}_1 \, d\tilde{x}_2 + O(a^{k+1/2}). \qquad [3.38]$$

Evaluate now the force $\mathbf{F}_A$ and torque $\mathbf{G}_A$ resulting from the flow field $(\tilde{\mathbf{u}}_A, \tilde{p}_A)$ produced by the direct approach of the two surfaces W and W'. From [3.3] and [3.4] it is seen that the values of $(\tilde{u}_A)_1$ and $(\tilde{u}_A)_2$ are given by

$$(\tilde{u}_A)_1 = \frac{1}{2\mu}(\tilde{x}_3^2 - \tilde{x}_3 h)\left(\frac{\partial\tilde{p}_A}{\partial\tilde{x}_1}\right), \qquad (\tilde{u}_A)_2 = \frac{1}{2\mu}(\tilde{x}_3^2 - \tilde{x}_3 h)\left(\frac{\partial\tilde{p}_A}{\partial\tilde{x}_2}\right). \qquad [3.39]$$

From the value of $\tilde{p}_A$ given by [3.24] it is seen that if $\tilde{x}_1$ and $\tilde{x}_2$ are replaced by $-\tilde{x}_1$ and $-\tilde{x}_2$ respectively, the value of $\tilde{p}_A$ is unchanged whereas $(\tilde{u}_A)_1$ and $(\tilde{u}_A)_2$ become $-(\tilde{u}_A)_1$ and $-(\tilde{u}_A)_2$ respectively. Hence it is seen that the values of $\mathbf{F}_A$ and $\mathbf{G}_A$ determined from [3.33], [3.34], [3.37] and [3.38] with $k = 0$ are

$$(F_A)_1 = O(\ln a), \qquad (F_A)_2 = O(\ln a), \qquad (F_A)_3 = a^{-1}\int_{S_\varepsilon} (-\tilde{p}_A) \, d\tilde{x}_1 \, d\tilde{x}_2 + O(a^{-1/2}) \qquad [3.40]$$

and

$$(G_A)_1 = O(\ln a), \qquad (G_A)_2 = O(\ln a),$$

$$(G_A)_3 = \int_{S_\varepsilon} \{[1/R_1 - 1/R_2]\tilde{x}_1\tilde{x}_2\tilde{p}_A + \mu[\tilde{x}_1(\partial(\tilde{u}_A)_2/\partial\tilde{x}_3) - \tilde{x}_2(\partial(\tilde{u}_A)_1/\partial\tilde{x}_3)]\} \, d\tilde{x}_1 \, d\tilde{x}_2 + f(\varepsilon) + O(a^{1/2}) \qquad [3.41]$$

where it has been noted that the integrals in [3.33] and [3.34] over the area Σ_ε are of order a^0 and may be functions of ε which tend to infinity as $\varepsilon \to 0$. Actually it would appear from [3.37] and [3.38] that $(F_A)_1$, $(F_A)_2$, $(G_A)_1$ and $(G_A)_2$ are of order a^0, but a closer examination of the error term of order $a^{1/2}$ in $\tilde{p}_A$ shows that it is of order $\hat{r}^{-3}$ as $\hat{r} \to \infty$, so that the integrals in the expressions for $(F_A)_1$, $(F_A)_2$, $(G_A)_1$ and $(G_A)_2$ possess logarithmic singularities and so give rise to terms of order $\ln a$ (in much the same manner as the expression for $(G_A)_3$ yields a term of order $\ln a$ as shown below).

By making use of [3.39] for $(\tilde{u}_A)_1$ and $(\tilde{u}_A)_2$, the above equation for $(G_A)_3$ reduces to

$$(G_A)_3 = \int_{S_\varepsilon} \{(1/R_1 - 1/R_2)\tilde{x}_1\tilde{x}_2\tilde{p}_A + \tfrac{1}{2}h[\tilde{x}_2(\partial\tilde{p}_A/\partial\tilde{x}_1) - \tilde{x}_1(\partial\tilde{p}_A/\partial\tilde{x}_2)]\}\, d\tilde{x}_1\, d\tilde{x}_2 + f(\varepsilon) + O(a^{1/2}). \qquad [3.42]$$

From [2.18] and [3.17], the relation between the $\tilde{x}_1$, $\tilde{x}_2$ variables and the $\hat{x}_1$, $\hat{x}_2$ variables is

$$\tilde{\mathbf{x}} = \mathbf{A} \cdot \hat{\mathbf{x}} \qquad [3.43]$$

where $\mathbf{A}$ is the orthogonal matrix

$$\mathbf{A} = (\bar{\mathbf{x}}_1^*, \bar{\mathbf{x}}_2^*),$$

$\bar{\mathbf{x}}_1^*$ and $\bar{\mathbf{x}}_2^*$ being the normalized eigenvectors of the matrix $\mathbf{K}$. Since $\mathbf{A}$ is orthogonal

$$\mathbf{A} = \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix} \qquad [3.44]$$

so that the eigenvectors $\bar{\mathbf{x}}_1^*$ and $\bar{\mathbf{x}}_2^*$ are

$$\bar{\mathbf{x}}_1^* = \begin{pmatrix} \cos\chi \\ -\sin\chi \end{pmatrix} \qquad \bar{\mathbf{x}}_2^* = \begin{pmatrix} \sin\chi \\ \cos\chi \end{pmatrix}. \qquad [3.45]$$

Hence the transformation [3.43] is

$$\tilde{x}_1 = \hat{x}_1 \cos\chi + \hat{x}_2 \sin\chi, \qquad \tilde{x}_2 = -\hat{x}_1 \sin\chi + \hat{x}_1 \cos\chi. \qquad [3.46]$$

Substituting these values into [3.40] and [3.42] for $(F_A)_3$ and $(G_A)_3$, one obtains

$$(F_A)_3 = -a^{-1}\int_{S_\varepsilon} \tilde{p}_A\, d\hat{x}_1\, d\hat{x}_2 + O(a^{-1/2})$$

$$(G_A)_3 = \int_{S_\varepsilon} \{(-\sin\chi\cos\chi)\hat{x}_1^2 + (\cos^2\chi - \sin^2\chi)\hat{x}_1\hat{x}_2$$

$$+ (\sin\chi\cos\chi)\hat{x}_2^2\}(1/R_1 - 1/R_2)\tilde{p}_A\, d\hat{x}_1\, d\hat{x}_2$$

$$+ \int_{S_\varepsilon} \tfrac{1}{2}h[\hat{x}_2(\partial\tilde{p}_A/\partial\hat{x}_1) - \hat{x}_1(\partial\tilde{p}_A/\partial\hat{x}_2)]\, d\hat{x}_1\, d\hat{x}_2 + f(\varepsilon) + O(a^{1/2}). \qquad [3.47]$$

A further substitution using [3.21] yields

$$(F_A)_3 = -a^{-1}(\lambda_1\lambda_2)^{-1/2}\int_{\hat{r}=0}^{a^{-1/2}\varepsilon}\int_{\theta=0}^{2\pi} \tilde{p}_A\hat{r}\, d\hat{r}\, d\theta + O(a^{-1/2})$$

$$(G_A)_3 = \int_{\hat{r}=0}^{a^{-1/2}\varepsilon} \int_{\theta=0}^{2\pi} \left[\frac{(-\sin\chi\cos\chi)}{\lambda_1}\cos^2\theta + (\cos^2\chi - \sin^2\chi)(\lambda_1\lambda_2)^{-1/2}\sin\theta\cos\theta + \frac{(\sin\chi\cos\chi)}{\lambda_2}\sin^2\theta \right](1/R_1 - 1/R_2)\tilde{p}_A(\hat{r}^3\,\mathrm{d}\hat{r}\,\mathrm{d}\theta)(\lambda_1\lambda_2)^{-1/2}\,\mathrm{d}\hat{r}\,\mathrm{d}\theta$$

$$+ \int_{\hat{r}=0}^{a^{-1/2}\varepsilon} \int_{\theta=0}^{2\pi} \tfrac{1}{2}\{(\lambda_1/\lambda_2)^{1/2}[\hat{r}\sin\theta\cos\theta(\partial\tilde{p}_A/\partial\hat{r}) - \sin^2\theta(\partial\tilde{p}_A/\partial\theta)]$$

$$- (\lambda_2/\lambda_1)^{1/2}[\hat{r}\sin\theta\cos\theta(\partial\tilde{p}_A/\partial\hat{r}) + \cos^2\theta(\partial\tilde{p}_A/\partial\theta)](1 + \hat{r}^2)\hat{r}(\lambda_1\lambda_2)^{-1/2}\}\,\mathrm{d}\hat{r}\,\mathrm{d}\theta$$

$$+ f(\varepsilon) + O(a^{1/2}). \qquad [3.48]$$

If the value of $\tilde{p}_A$ from [3.24] is substituted into the above equations and the integration with respect to θ performed, the values of $(F_A)_3$ and $(G_A)_3$ are obtained as

$$(F_A)_3 = 6\pi(\mu/a)(U'_3 - U_3)(\lambda_1 + \lambda_2)^{-1}(\lambda_1\lambda_2)^{-1/2}\int_0^{a^{-1/2}\varepsilon} \hat{r}(1 + \hat{r}^2)^{-2}\,\mathrm{d}\hat{r} + O(a^{-1/2}) \qquad [3.49]$$

and

$$(G_A)_3 = \mu\,3\pi(U'_3 - U_3)(\lambda_1 + \lambda_2)^{-1}(\lambda_1\lambda_2)^{-1/2}(1/R_1 - 1/R_2)(1/\lambda_1 - 1/\lambda_2)\sin\chi\cos\chi \int_0^{a^{-1/2}\varepsilon} \hat{r}^3(1 + \hat{r}^2)^{-2}\,\mathrm{d}\hat{r} + f(\varepsilon) + O(a^{1/2}). \qquad [3.50]$$

As $a \to 0$, the integral in [3.49] tends to

$$\int_0^{\infty} \hat{r}(1 + \hat{r}^2)^{-2}\,\mathrm{d}\hat{r} = \tfrac{1}{2}. \qquad [3.51]$$

However, one cannot replace the upper limit in the integral in [3.50] by infinity since it then becomes divergent. Straightforward integration yields the asymptotic form of this integral for $a \to 0$ as

$$\int_0^{a^{-1/2}\varepsilon} \hat{r}^3(1 + \hat{r}^2)^{-2}\,\mathrm{d}\hat{r} = -\tfrac{1}{2}\ln a + (\ln\varepsilon - \tfrac{1}{2}) + O(a). \qquad [3.52]$$

The substitution of this integral into [3.50] shows that $(G_A)_3$ contains a term in $(\ln a)$ and a term in (a^0). This term in a^0 has a contribution from the inner area S_ε proportional to $(\ln\varepsilon)$. However since the value of $(G_A)_3$ as calculated must be independent of ε in the limit of $\varepsilon \to 0$, it follows that the contribution proportional to $(\ln\varepsilon)$ in the term of order (a^0) must be exactly cancelled by the contribution $f(\varepsilon)$ from the outer region Σ_ε. Therefore one has $\mathbf{F}_A$ and $\mathbf{G}_A$ given by

$$(F_A)_1 = O(\ln a), \qquad (F_A)_2 = O(\ln a),$$

$$(F_A)_3 = 3\pi(\mu/a)(U'_3 - U_3)(\lambda_1 + \lambda_2)^{-1}(\lambda_1\lambda_2)^{-1/2} + O(a^{-1/2}) \qquad [3.53a]$$

and

$(G_A)_1 = O(\ln a), \qquad (G_A)_2 = O(\ln a),$

$$(G_A)_3 = -\mu(\ln a)(3\pi/2)(U_3' - U_3)(\lambda_1 + \lambda_2)^{-1}(\lambda_1\lambda_2)^{-1/2}(1/R_1 - 1/R_2)(1/\lambda_1 - 1/\lambda_2)\sin\gamma\cos\gamma + O(a^0). \qquad [3.54]$$

If the surfaces W and W' are each locally symmetric so that replacement of x_1 and x_2 by $-x_1$ and $-x_2$ leaves x_3 and x_3^* unchanged, the form of the surfaces are then given by [2.2b] and [2.3b]. The theory is then altered only with respect to the order of magnitude of the error terms involved (e.g. $\tilde{p}_A$ given by [3.24] would now possess an error of order a^1 instead of order $a^{1/2}$) and one would obtain that $(F_A)_1$, $(F_A)_2$, $(G_A)_1$ and $(G_A)_2$ are no longer singular being of order a^0. Also $(F_A)_3$ and $(G_A)_3$ would still be given by [3.53a] and [3.54] except that $(F_A)_3$ now has an error term of order $(\ln a)$ instead of $a^{-1/2}$ as was shown by Cox & Brenner (1967) and by Cooley & O'Neill (1969). Thus, for this case,

$$(F_A)_3 = 3\pi(\mu/a)(U_3' - U_3)(\lambda_1 + \lambda_2)^{-1}(\lambda_1\lambda_2)^{-1/2} + O(\ln a). \qquad [3.53b]$$

4. TANGENTIAL AND ROLLING MOTION OF SURFACES

In this section we consider the effect of the flow field $(\tilde{\mathbf{u}}_B, p_B)$ resulting from the tangential and rolling motion of the surfaces W and W'. Such a flow is that due to the components U_1, U_2, U_1' and U_2' of the velocities $\mathbf{U}$ and $\mathbf{U}'$ and that due to the components Ω_1, Ω_2, Ω_1' and Ω_2' of the angular velocities $\mathbf{\Omega}$ and $\mathbf{\Omega}'$. Taking $k = \frac{1}{2}$, the boundary conditions [2.28] and [2.29] for $(\tilde{\mathbf{u}}_B, \tilde{p}_B)$ become for small values of a,

$$(\tilde{u}_B)_1 = U_1 + O(a), \quad (\tilde{u}_B)_2 = U_2 + O(a), \quad (\tilde{u}_B)_3 = \Omega_1\bar{x}_2 - \Omega_2\bar{x}_1 \quad \text{on} \quad \bar{x}_3 = 0 \qquad [4.1]$$

$$(\tilde{u}_B)_1 = U_1' + O(a), \quad (\tilde{u}_B)_2 = U_2' + O(a), \quad (\tilde{u}_B)_3 = \Omega_1'\bar{x}_2 - \Omega_2'\bar{x}_1 \quad \text{on} \quad \bar{x}_3 = h \qquad [4.2]$$

where $\bar{x}_3 = 0$, $\bar{x}_3 = h$ for the walls are correct to order $a^{1/2}$.

Since $(\tilde{\mathbf{u}}_B, \tilde{p}_B)$ satisfies [2.21] it may readily be shown in a manner similar to that used for $(\tilde{\mathbf{u}}_A, \tilde{p}_A)$ [see § 3], the value of the flow-field $(\tilde{\mathbf{u}}_B, \tilde{p}_B)$ is to order (a^{+1}) given by

$$\tilde{p}_B = \tilde{p}_B(\bar{x}_1, \bar{x}_2)$$

$$(\tilde{u}_B)_1 = \frac{1}{2\mu}(\partial\tilde{p}_B/\partial\bar{x}_1)\bar{x}_3^2 + A\bar{x}_3 + U_1, \quad (\tilde{u}_B)_2 = \frac{1}{2\mu}(\partial\tilde{p}_B/\partial\bar{x}_2)\bar{x}_3^2 + B\bar{x}_3 + U_2,$$

$$\begin{aligned}(\tilde{u}_B)_3 = &-\frac{1}{6\mu}[(\partial^2\tilde{p}_B/\partial\bar{x}_1^2) + (\partial^2\tilde{p}_B/\partial\bar{x}_2^2)]\bar{x}_3^3 - \tfrac{1}{2}[(\partial A/\partial\bar{x}_1 + (\partial B/\partial\bar{x}_2)]\bar{x}_3^2\\ &-\frac{1}{2\mu}[\bar{x}_1/R_1(\partial\tilde{p}_B/\partial\bar{x}_1) + \bar{x}_2/R_2(\partial\tilde{p}_B/\partial\bar{x}_2)]\bar{x}_3^2 - ((A/R_1)\bar{x}_1 + (B/R_2)\bar{x}_2)\bar{x}_3\\ &+ (\Omega_1\bar{x}_2 - \Omega_2\bar{x}_1)\end{aligned} \qquad [4.3]$$

where the quantities A and B are functions of $\bar{x}_1$ and $\bar{x}_2$ only and are given by

$$A = [(U_1' - U_1)/h] - \frac{1}{2\mu}(\partial\tilde{p}_B/\partial\bar{x}_1)h, \qquad B = [(U_2' - U_2)/h] - \frac{1}{2\mu}(\partial\tilde{p}_B/\partial\bar{x}_2)h. \qquad [4.4]$$

By substituting the value of $(\tilde{u}_B)_3$ given by [4.3] into the last boundary condition of [4.2] one obtains an equation for $\tilde{p}_B$ which may be reduced to

$$\begin{aligned}-(1/12\mu)h^3\bar{\nabla}^2\tilde{p}_B + \frac{1}{4\mu}h^2(\bar{\nabla}\tilde{p}_B\cdot\bar{\nabla}h) = &\{[(U_1' - U_1)/R_1] - (\Omega_2' - \Omega_2)\}\bar{x}_1\\ + \{[(U_2' - U_2)/R_2] + (\Omega_1' - \Omega_1)\}\bar{x}_2 &- \tfrac{1}{2}(U_1' - U_1)\partial h/\partial\bar{x}_1 - \tfrac{1}{2}(U_2' - U_2)\partial h/\partial\bar{x}_2\end{aligned} \qquad [4.5]$$

where differentiation is with respect to the variables $\bar{x}_1$ and $\bar{x}_2$.

Substituting into the above expression the value of $h(\bar{x}_1, \bar{x}_2)$ given by [2.2], one obtains

$$(1/\mu)\nabla \cdot (h^3 \nabla \tilde{p}_B) = \mathbf{q}^T \cdot \bar{\mathbf{x}} \qquad [4.6]$$

where $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2)^T$ and $\mathbf{q}$ is a vector defined by

$$\mathbf{q} = 12\begin{pmatrix} -(\Omega_2' - \Omega_2) + (U_1' - U_1)\left(\dfrac{1}{2R_1} - \dfrac{\cos^2\phi}{2S_1} - \dfrac{\sin^2\phi}{2S_2}\right) - (U_2' - U_2)\dfrac{\sin\phi\cos\phi}{2}\left(\dfrac{1}{S_1} - \dfrac{1}{S_2}\right) \\ +(\Omega_1' - \Omega_1) + (U_2' - U_2)\left(\dfrac{1}{2R_2} - \dfrac{\sin^2\phi}{2S_1} - \dfrac{\cos^2\phi}{2S_2}\right) - (U_1' - U_1)\dfrac{\sin\phi\cos\phi}{2}\left(\dfrac{1}{S_1} - \dfrac{1}{S_2}\right) \end{pmatrix}. \qquad [4.7]$$

Changing to $\hat{x}_1$ and $\hat{x}_2$ variables by means of the transformation [3.17], [4.6] becomes

$$\frac{1}{\mu}\left\{\frac{\partial}{\partial \hat{x}_1}\left[(1 + \lambda_1\hat{x}_1^2 + \lambda_2\hat{x}_2^2)^3 \frac{\partial \tilde{p}_B}{\partial \hat{x}_1}\right] + \frac{\partial}{\partial \hat{x}_2}\left[(1 + \lambda_1\hat{x}_1^2 + \lambda_2\hat{x}_2^2)^3 \frac{\partial \tilde{p}_B}{\partial \hat{x}_2}\right]\right\} = (Q_1\hat{x}_1 + Q_2\hat{x}_2) \qquad [4.8]$$

where $\mathbf{Q} = (Q_1, Q_2)$ is a vector defined by

$$\mathbf{Q}^T = \mathbf{q}^T \cdot \mathbf{A} \quad \text{or} \quad \mathbf{Q} = \mathbf{A}^T \cdot \mathbf{q} \qquad [4.9]$$

and so by [3.44] for the matrix $\mathbf{A}$,

$$Q_1 = q_1 \cos\chi - q_2 \sin\chi, \qquad Q_2 = q_1 \sin\chi + q_2 \cos\chi. \qquad [4.10]$$

By making the further substitution [3.21], [4.8] may be written in terms of the $\hat{r}$ and θ variables as

$$\begin{aligned}&(\lambda_1\cos^2\theta + \lambda_2\sin^2\theta)\hat{r}^2\frac{\partial^2\tilde{p}_B}{\partial\hat{r}^2} + 2(\lambda_2 - \lambda_1)\sin\theta\cos\theta\,\hat{r}\frac{\partial^2\tilde{p}_B}{\partial\hat{r}\,\partial\theta} \\ &\quad + (\lambda_1\sin^2\theta + \lambda_2\cos^2\theta)\frac{\partial^2\tilde{p}_B}{\partial\theta^2} + (\lambda_1\sin^2\theta + \lambda_2\cos^2\theta)\hat{r}\frac{\partial\tilde{p}_B}{\partial\hat{r}} \\ &\quad + 2(\lambda_1 - \lambda_2)\sin\theta\cos\theta\frac{\partial\tilde{p}_B}{\partial\theta} + \left(\frac{6\hat{r}^3}{1+\hat{r}^2}\right)(\lambda_1\cos^2\theta + \lambda_2\sin^2\theta)\frac{\partial\tilde{p}_B}{\partial\hat{r}} \\ &+ \left(\frac{6\hat{r}^2}{1+\hat{r}^2}\right)(\lambda_2 - \lambda_1)\sin\theta\cos\theta\frac{\partial\tilde{p}_B}{\partial\theta} = \left(\frac{Q_1}{(\lambda_1)^{1/2}}\cos\theta + \frac{Q_2}{(\lambda_2)^{1/2}}\sin\theta\right)\frac{\mu\hat{r}^3}{(1+\hat{r}^2)^3}. \end{aligned} \qquad [4.11]$$

It will be seen later that one only requires the form of $\tilde{p}_B$ for large values of $\hat{r}$ so the limiting form of [4.11] for $\hat{r} \to \infty$ may be taken, i.e.

$$\begin{aligned}&(\lambda_1\cos^2\theta + \lambda_2\sin^2\theta)\hat{r}^2\frac{\partial^2\tilde{p}_B}{\partial\hat{r}^2} + 2(\lambda_2 - \lambda_1)\sin\theta\cos\theta\,\hat{r}\frac{\partial^2\tilde{p}_B}{\partial\hat{r}\,\partial\theta} \\ &\quad + (\lambda_1\sin^2\theta + \lambda_2\cos^2\theta)\frac{\partial^2\tilde{p}_B}{\partial\theta^2} + \{(6\lambda_1 + \lambda_2)\cos^2\theta + (\lambda_1 + 6\lambda_2)\sin^2\theta\}\hat{r}\frac{\partial\tilde{p}_B}{\partial\hat{r}}\end{aligned}$$

$$+ 4(\lambda_2 - \lambda_1)\sin\theta\cos\theta\frac{\partial\tilde{p}_B}{\partial\theta} = \mu\hat{r}^{-3}\left(\frac{Q_1}{(\lambda_1)^{1/2}}\cos\theta + \frac{Q_2}{(\lambda_2)^{1/2}}\sin\theta\right). \quad [4.12]$$

In a manner similar to that for $\tilde{p}_A$ (see §3), it may be shown that for satisfactory matching onto the outer expansion one requires $\tilde{p}_B$ to satisfy

$$\tilde{p}_B = O(r^{-3}) \quad \text{as} \quad r \to \infty. \quad [4.13]$$

It may readily be shown that the solution of [4.12] satisfying [4.13] is

$$\tilde{p}_B = -\frac{1}{2}\mu\hat{r}^{-3}\left\{\frac{Q_1}{\lambda_1^{1/2}(3\lambda_1 + 2\lambda_2)}\cos\theta + \frac{Q_2}{\lambda_2^{1/2}(2\lambda_1 + 3\lambda_2)}\sin\theta\right\}. \quad [4.14a]$$

Due to the approximation made in [4.12] that $\hat{r}$ was very large, [4.14a] for $\tilde{p}_B$ gives really the first term in the asymptotic expansion of $\tilde{p}_B$ for large $\hat{r}$. Also since the walls given in [4.1] and [4.2] possess errors of order $a^{1/2}$, so does $\tilde{p}_B$. Thus

$$\tilde{p}_B = -\frac{1}{2}\mu\hat{r}^{-3}\left\{\frac{Q_1}{\lambda_1^{1/2}(3\lambda_1 + 2\lambda_2)}\cos\theta + \frac{Q_2}{\lambda_2^{1/2}(2\lambda_1 + 3\lambda_2)}\sin\theta\right\} + O(\hat{r}^{-2}) + O(a^{1/2}). \quad [4.14b]$$

By replacing $\hat{x}_1$ and $\hat{x}_2$ by $-\hat{x}_1$ and $-\hat{x}_2$ respectively or equivalently θ by $(\pi + \theta)$, the value of $\tilde{p}_B$ becomes $-\tilde{p}_B$. Also such a substitution into [4.3] shows that $(\tilde{u}_B)_1$ and $(\tilde{u}_B)_2$ remain unchanged. By making use of these symmetry properties of the flow, the force $\mathbf{F}_B$ and torque $\mathbf{G}_B$ on W resulting from the flow $(\tilde{\mathbf{u}}_B, \tilde{p}_B)$ may be calculated from [3.33], [3.34], [3.37] and [3.38] as

$$(F_B)_1 = \int_{S_\varepsilon}\left(-\frac{\tilde{x}_1\tilde{p}_B}{R_1} + \mu\frac{\partial(\tilde{u}_B)_1}{\partial\tilde{x}_3}\right)\mathrm{d}\tilde{x}_1\,\mathrm{d}\tilde{x}_2 + f_1(\varepsilon) + O(a^{1/2}),$$

$$(F_B)_2 = \int_{S_\varepsilon}\left(-\frac{\tilde{x}_2\tilde{p}_B}{R_2} + \mu\frac{\partial(\tilde{u}_B)_2}{\partial\tilde{x}_3}\right)\mathrm{d}\tilde{x}_1\,\mathrm{d}\tilde{x}_2 + f_2(\varepsilon) + O(a^{1/2}),$$

$$(F_B)_3 = O(\ln a) \quad [4.15]$$

and

$$(G_B)_1 = \int_{S_\varepsilon}(-\tilde{x}_2\tilde{p}_B)\,\mathrm{d}\tilde{x}_1\,\mathrm{d}\tilde{x}_2 + g_1(\varepsilon) + O(a^{1/2}),$$

$$(G_B)_2 = \int_{S_\varepsilon}(+\tilde{x}_1\tilde{p}_B)\,\mathrm{d}\tilde{x}_1\,\mathrm{d}\tilde{x}_2 + g_2(\varepsilon) + O(a^{1/2}),$$

$$(G_B)_3 = O(a^0). \quad [4.16]$$

That $(F_B)_3$ is of order ln a rather than a^0 follows from the fact that the error term of order $a^{1/2}$ in $\tilde{p}_B$ behaves like $\hat{r}^{-2}$ as $\hat{r} \to \infty$ giving a logarithmic singularity for the integral for $(F_B)_3$. Substituting the values of $(\tilde{u}_B)_1$ and $(\tilde{u}_B)_2$ into these expressions from [4.3] and changing to the $\hat{x}_1$ and $\hat{x}_2$ variables, one obtains

$$(F_B)_1 = \int_{S_\varepsilon}\left[-\frac{\tilde{p}_B}{R_1}(\hat{x}_1\cos\chi + \hat{x}_2\sin\chi) + \frac{\mu(U_1' - U_1)}{h}\right.$$

$$-\frac{1}{2}h\left(\cos\chi\frac{\partial\tilde{p}_B}{\partial\hat{x}_1}+\sin\chi\frac{\partial\tilde{p}_B}{\partial\hat{x}_2}\right)\Bigg]\mathrm{d}\hat{x}_1\,\mathrm{d}\hat{x}_2+f_1(\varepsilon)+O(a^{1/2}),$$

$$(F_B)_2=\int_{S_\varepsilon}\Bigg[-\frac{\tilde{p}_B}{R_2}(-\hat{x}_1\sin\chi+\hat{x}_2\cos\chi)+\frac{\mu(U_2'-U_2)}{h}$$

$$-\frac{1}{2}h\left(-\sin\chi\frac{\partial\tilde{p}_B}{\partial\hat{x}_1}+\cos\chi\frac{\partial\tilde{p}_B}{\partial\hat{x}_2}\right)\Bigg]\mathrm{d}\hat{x}_1\,\mathrm{d}\hat{x}_2+f_2(\varepsilon)+O(a^{1/2}),$$

$$(F_B)_3=O(\ln a). \qquad [4.17]$$

and

$$(G_B)_1=\int_{S_\varepsilon}-\tilde{p}_B(-\hat{x}_1\sin\chi+\hat{x}_2\cos\chi)\,\mathrm{d}\hat{x}_1\,\mathrm{d}\hat{x}_2+g_1(\varepsilon)+O(a^{1/2}),$$

$$(G_B)_2=\int_{S_\varepsilon}+\tilde{p}_B(\hat{x}_1\cos\chi+\hat{x}_2\sin\chi)\,\mathrm{d}\hat{x}_1\,\mathrm{d}\hat{x}_2+g_2(\varepsilon)+O(a^{1/2}),$$

$$(G_B)_3=O(a^0). \qquad [4.18]$$

These integrals may be evaluated by changing from $\hat{x}_1$ and $\hat{x}_2$ to $\hat{r}$ and θ variables and by substituting the value of $\tilde{p}_B$ from [4.14b]. The region of integration S_ε in terms of the $\hat{r}$ and θ variables is

$$0\le\hat{r}\le a^{-1/2}\varepsilon \qquad 0\le\theta\le 2\pi \qquad [4.19]$$

so that upon performing the θ integration the above expressions for $\mathbf{F}_B$ and $\mathbf{G}_B$ become

$$(F_B)_1=\frac{1}{2}\pi\mu\int_0^{a^{-1/2}\varepsilon}\Bigg[\frac{\hat{r}^{-1}}{R_1}\left\{\frac{Q_1\cos\chi}{\lambda_1(3\lambda_1+2\lambda_2)}+\frac{Q_2\sin\chi}{\lambda_2(2\lambda_1+3\lambda_2)}\right\}+\frac{4(U_1'-U_1)\hat{r}}{(1+\hat{r}^2)}-$$

$$\hat{r}^{-3}(1+\hat{r}^2)\left\{\frac{Q_1\cos\chi}{(3\lambda_1+2\lambda_2)}+\frac{Q_2\sin\chi}{(2\lambda_1+3\lambda_2)}\right\}\Bigg](\lambda_1\lambda_2)^{-1/2}\,\mathrm{d}\hat{r}+f_1(\varepsilon)+O(a^{1/2}),$$

$$(F_B)_2=\frac{1}{2}\pi\mu\int_0^{a^{-1/2}\varepsilon}\Bigg[\frac{\hat{r}^{-1}}{R_2}\left\{-\frac{Q_1\sin\chi}{\lambda_1(3\lambda_1+2\lambda_2)}+\frac{Q_2\cos\chi}{\lambda_2(2\lambda_1+3\lambda_2)}\right\}+\frac{4(U_2'-U_2)\hat{r}}{(1+\hat{r}^2)}-$$

$$\hat{r}^{-3}(1+\hat{r}^2)\left\{-\frac{Q_1\sin\chi}{(3\lambda_1+2\lambda_2)}+\frac{Q_2\cos\chi}{(2\lambda_1+3\lambda_2)}\right\}\Bigg](\lambda_1\lambda_2)^{-1/2}\,\mathrm{d}\hat{r}+f_2(\varepsilon)+O(a^{1/2}),$$

$$(F_B)_3=O(\ln a), \qquad [4.20]$$

and

$$(G_B)_1=\frac{1}{2}\pi\mu\left\{-\frac{Q_1\sin\chi}{\lambda_1(3\lambda_1+2\lambda_2)}+\frac{Q_2\cos\chi}{\lambda_2(2\lambda_1+3\lambda_2)}\right\}(\lambda_1\lambda_2)^{-1/2}\int_0^{a^{-1/2}\varepsilon}\hat{r}^{-1}\,\mathrm{d}\hat{r}$$

$$+g_1(\varepsilon)+O(a^{1/2}),$$

$$(G_B)_2=-\frac{1}{2}\pi\mu\left\{\frac{Q_1\cos\chi}{\lambda_1(3\lambda_1+2\lambda_2)}+\frac{Q_2\sin\chi}{\lambda_2(2\lambda_1+3\lambda_2)}\right\}(\lambda_1\lambda_2)^{-1/2}\int_0^{a^{-1/2}\varepsilon}\hat{r}^{-1}\,\mathrm{d}\hat{r}$$

$$+ g_2(\varepsilon) + O(a^{1/2}),$$

$$(G_B)_3 = O(a^0). \qquad [4.21]$$

Actually, the term $\hat{r}^{-1}$ in the integrands in the above equations are correct only in the limit $\hat{r} \to \infty$ (since $\tilde{p}_B$ was obtained from [4.12] rather than the complete [4.11]) and they really have no singularity at $\hat{r} = 0$. However the logarithmic singularity at $\hat{r} \to \infty$, allows one to proceed as for the calculation of $(G_A)_3$ given in §3, to obtain the asymptotic forms of $\mathbf{F}_B$ and $\mathbf{G}_B$ as

$$(F_B)_1 = -\mu(\ln a)\frac{\pi}{4(\lambda_1\lambda_2)^{1/2}}\left[\frac{Q_1\cos\chi}{(3\lambda_1 + 2\lambda_2)}\left(\frac{1}{R_1\lambda_1} - 1\right) + \frac{Q_2\sin\chi}{(2\lambda_1 + 3\lambda_2)}\left(\frac{1}{R_1\lambda_2} - 1\right) + 4(U_1' - U_1)\right] + O(a^0),$$

$$(F_B)_2 = -\mu(\ln a)\frac{\pi}{4(\lambda_1\lambda_2)^{1/2}}\left[-\frac{Q_1\sin\chi}{(3\lambda_1 + 2\lambda_2)}\left(\frac{1}{R_2\lambda_1} - 1\right) + \frac{Q_2\cos\chi}{(2\lambda_1 + 3\lambda_2)}\left(\frac{1}{R_2\lambda_2} - 1\right) + 4(U_2' - U_2)\right] + O(a^0), \qquad [4.22]$$

$$(F_B)_3 = O(\ln a).$$

and
$$(G_B)_1 = -\mu(\ln a)\frac{\pi}{4(\lambda_1\lambda_2)^{1/2}}\left\{-\frac{Q_1\sin\chi}{\lambda_1(3\lambda_1 + 2\lambda_2)} + \frac{Q_2\cos\chi}{\lambda_2(2\lambda_1 + 3\lambda_2)}\right\} + O(a^0),$$

$$(G_B)_2 = +\mu(\ln a)\frac{\pi}{4(\lambda_1\lambda_2)^{1/2}}\left\{+\frac{Q_1\cos\chi}{\lambda_1(3\lambda_1 + 2\lambda_2)} + \frac{Q_2\sin\chi}{\lambda_2(2\lambda_1 + 3\lambda_2)}\right\} + O(a^0),$$

$$(G_B)_3 = O(a^0). \qquad [4.23]$$

If the surfaces W and W' are locally symmetric so that they are given by [2.2b] and [2.3b] then $\mathbf{F}_B$ and $\mathbf{G}_B$ are as given above except that $(F_B)_3$ is no longer singular, being of order a^0. The quantities Q_1 and Q_2 in [4.22] and [4.23] are functions of U_1, U_2, U_1', U_2', Ω_1, Ω_2, Ω_1' and Ω_2' and are given by [4.10] and [4.7], i.e.

$$Q_1 = q_1\cos\chi - q_2\sin\chi, \qquad Q_2 = q_1\sin\chi + q_2\cos\chi \qquad [4.24]$$

where
$$q_1 = 12\left\{-(\Omega_2' - \Omega_2) + (U_1' - U_1)\left(\frac{1}{2R_1} - \frac{\cos^2\phi}{2S_1} - \frac{\sin^2\phi}{2S_2}\right) - (U_2' - U_2)\frac{\sin\phi\cos\phi}{2}\left(\frac{1}{S_1} - \frac{1}{S_2}\right)\right\},$$

$$q_2 = 12\left\{+(\Omega_1' - \Omega_1) + (U_2' - U_2)\left(\frac{1}{2R_2} - \frac{\sin^2\phi}{2S_1} - \frac{\cos^2\phi}{2S_2}\right)\right.$$

$$-(U_1' - U_1)\frac{\sin\phi\cos\phi}{2}\left(\frac{1}{S_1} - \frac{1}{S_2}\right)\bigg\}. \qquad [4.25]$$

5. ROTATIONAL MOTION OF SURFACES ABOUT NORMAL

The flow field $(\tilde{\mathbf{u}}_C, \tilde{p}_C)$ resulting from the rotational motion of the surfaces about their mutual normal is now considered. Such a flow field is that produced by the components Ω_3 and Ω_3' of $\mathbf{\Omega}$ and $\mathbf{\Omega}'$, there being no translational motion of the surfaces. Taking $k = 1$, the boundary conditions [2.30] and [2.31] for $(\tilde{\mathbf{u}}_C, \tilde{p}_C)$ become

$$(\tilde{u}_C)_1 = -\Omega_3\bar{x}_2, \quad (\tilde{u}_C)_2 = +\Omega_3\bar{x}_1, \quad (\tilde{u}_C)_3 = 0, \quad \text{on } \bar{x}_3 = 0, \qquad [5.1]$$

and $$(\tilde{u}_C)_1 = -\Omega_3'\bar{x}_2, \quad (\tilde{u}_C)_2 = +\Omega_3'\bar{x}_1, \quad (\tilde{u}_C)_3 = 0, \quad \text{on } \bar{x}_3 = h(\bar{x}_1, \bar{x}_2). \qquad [5.2]$$

Since $(\tilde{\mathbf{u}}_C, \tilde{p}_C)$ satisfies also [2.22], one may proceed as in §3 for $(\tilde{\mathbf{u}}_A, \tilde{p}_A)$ and in §4 for $(\mathbf{u}_B, p_B)$. Therefore one may obtain

$$\tilde{p}_C = \tilde{p}_C(\bar{x}_1, \bar{x}_2)$$

$$(\tilde{u}_C)_1 = (2\mu)^{-1}(\partial\tilde{p}_C/\partial\bar{x}_1)x_3^2 + A\bar{x}_3 - \Omega_3\bar{x}_2$$

$$(\tilde{u}_C)_2 = (2\mu)^{-1}(\partial\tilde{p}_C/\partial\bar{x}_2)\bar{x}_3^2 + B\bar{x}_3 + \Omega_3\bar{x}_1,$$

$$(\tilde{\mu}_C)_3 = -(6\mu)^{-1}(\partial^2\tilde{p}_C/\partial\bar{x}_1^2 + \partial^2\tilde{p}_C/\partial\bar{x}_2^2)\bar{x}_3^3 - \tfrac{1}{2}(\partial A/\partial\bar{x}_1 + \partial B/\partial\bar{x}_2)\bar{x}_3^2$$
$$-(2\mu)^{-1}(\bar{x}_1/R_1)\partial\tilde{p}_C/\partial\bar{x}_1 + (\bar{x}_2/R_2)\partial\tilde{p}_C/\partial\bar{x}_2)\bar{x}_3^2 - ((A/R_1)\bar{x}_1 + (B/R_2)\bar{x}_2)\bar{x}_3, \qquad [5.3]$$

where A and B are functions of $\bar{x}_1$ and $\bar{x}_2$ and are given by

$$A = -(2\mu)^{-1}(\partial\tilde{p}_C/\partial\bar{x}_1)h + (\Omega_3 - \Omega_3')\bar{x}_2/h, \qquad B = -(2\mu)^{-1}(\partial\tilde{p}_C/\partial\bar{x}_2)h - (\Omega_3 - \Omega_3')\bar{x}_1/h. \qquad [5.4]$$

By making use of the boundary condition [5.2] for $(\tilde{u}_C)_3$ on $\bar{x}_3 = h$, one obtains an equation for p_C which may be written as

$$(1/\mu)\bar{\nabla}\cdot(h^3\bar{\nabla}\tilde{p}_C) = (\Omega_3 - \Omega_3')\{6(\bar{x}_1(\partial h/\partial\bar{x}_2) - x_2(\partial h/\partial\bar{x}_1) + 12(1/R_1 - 1/R_2)\bar{x}_1\bar{x}_2\}. \qquad [5.5]$$

With respect to the $\hat{x}_1$ and $\hat{x}_2$ variables, this equation may be written as

$$1/\mu\{\partial/\partial\hat{x}_1[(1 + \lambda_1\hat{x}_1^2 + \lambda_2\hat{x}_2^2)^3(\partial p_C/\partial\hat{x}_1)] + \partial/\partial\hat{x}_2[(1 + \lambda_1\hat{x}_1^2 + \lambda_2\hat{x}_2^2)^3\partial p_C/\partial\hat{x}_2]\}$$
$$= 12(\Omega_3 - \Omega_3')[(\lambda_2 - \lambda_1)\hat{x}_1\hat{x}_2 + (1/R_1 - 1/R_2)\{-\hat{x}_1^2\sin\chi\cos\chi$$
$$+ \hat{x}_1\hat{x}_2(\cos^2\chi - \sin^2\chi) + \hat{x}_2^2\sin\chi\cos\chi\}]. \qquad [5.6]$$

By means of [3.21], this equation may be written in terms of $\hat{r}$ and θ as

$$(\lambda_1\cos^2\theta + \lambda_2\sin^2\theta)\hat{r}^2(\partial^2\tilde{p}_C/\partial\hat{r}^2) + 2(\lambda_2 - \lambda_1)\sin\theta\cos\theta\hat{r}(\partial^2\tilde{p}_C/\partial\hat{r}\,\partial\theta)$$
$$+ (\lambda_1\sin^2\theta + \lambda_2\cos^2\theta)(\partial^2\tilde{p}_C/\partial\theta^2) + (\lambda_1\sin^2\theta + \lambda_2\cos^2\theta)\hat{r}(\partial\tilde{p}_C/\partial\hat{r})$$
$$+ 2(\lambda_1 - \lambda_2)\sin\theta\cos\theta(\partial\tilde{p}_C/\partial\theta) + (6\hat{r}^3/1 + \hat{r}^2)(\lambda_1\cos^2\theta + \lambda_2\sin^2\theta)\partial\tilde{p}_C/\partial\hat{r}$$
$$+ (6\hat{r}^2/1 + \hat{r}^2)(\lambda_2 - \lambda_1)\sin\theta\cos\theta(\partial\tilde{p}_C/\partial\theta) = [12\mu\hat{r}^4/(1 + \hat{r}^2)^3](\Omega_3 - \Omega_3')$$
$$[(\lambda_2 - \lambda_1)\sin\theta\cos\theta/(\lambda_1\lambda_2)^{1/2} + (1/R_1 - 1/R_2)\{-(\sin\chi\cos\chi\cos^2\theta)/\lambda_1$$

$$+ [(\cos^2\chi - \sin^2\chi)\sin\theta\cos\theta]/(\lambda_1\lambda_2)^{1/2} + (\sin\chi\cos\chi\sin^2\theta)/\lambda_2\}]. \quad [5.7]$$

It may be shown that $\tilde{p}_C$ for satisfactory matching onto the outer expansion must satisfy the condition

$$\tilde{p}_C = O(\hat{r}^{-2}) \qquad \text{as } \hat{r} \to \infty. \quad [5.8]$$

By letting $\hat{r} \to \infty$ in [5.7], it may be shown that the first term in the asymptotic expansion of $\tilde{p}_C$ for large $\hat{r}$ is of the form

$$\tilde{p}_C = \mu\hat{r}^{-2}(A'\sin 2\theta + B'\cos 2\theta + C'), \quad [5.9]$$

where A', B' and C' are given by

$$A' = -\frac{(\Omega_3 - \Omega_3')}{(\lambda_1 + \lambda_2)}\left[\frac{(\lambda_2 - \lambda_1)}{(\lambda_1\lambda_2)^{1/2}} + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{(\cos^2\chi - \sin^2\chi)}{(\lambda_1\lambda_2)^{1/2}}\right],$$

$$B' = +\frac{(\Omega_3 - \Omega_3')}{\lambda_1\lambda_2}\left(\frac{1}{R_1} - \frac{1}{R_1}\right)\left\{\frac{(\lambda_1 - \lambda_2)^2}{2(\lambda_1 + \lambda_2)^2} + 1\right\}\sin\chi\cos\chi,$$

$$C' = -\frac{3(\Omega_3 - \Omega_3')}{2(\lambda_1 + \lambda_2)}\left(\frac{1}{R_1} - \frac{1}{R_1}\right)\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right)\sin\chi\cos\chi. \quad [5.10]$$

From [3.33], [3.34], [3.37] and [3.38] with $k = 1$, the value of the force $\mathbf{F}_C$ and torque $\mathbf{G}_C$ on W resulting from the flow $(\tilde{\mathbf{u}}_C, \tilde{p}_C)$ is

$$(F_C)_1 = O(a^0), \quad (F_C)_2 = O(a^0), \quad (F_C)_3 = -\int_{S_\varepsilon} \tilde{p}_C\, d\tilde{x}_1\, d\tilde{x}_2 + f(\varepsilon) + O(a^{+1}) \quad [5.11]$$

and

$$\mathbf{G}_C = O(a^0) \quad \text{as } a \to 0, \quad [5.12]$$

where $f(\varepsilon)$ may be singular as $\varepsilon \to 0$. Changing to $\hat{r}, \theta$ variables, the above expression for $(F_C)_3$ becomes

$$(F_C)_3 = -(\lambda_1\lambda_2)^{-1/2}\int_{\hat{r}=0}^{a^{-1/2}\varepsilon}\int_{\theta=0}^{2\pi} \tilde{p}_C\hat{r}\, d\hat{r}\, d\theta + f(\varepsilon) + O(a^{+1}). \quad [5.13]$$

Substituting the asymptotic value [5.9] of $\tilde{p}_C$ into this equation and performing the integration, one obtains

$$(F_C)_3 = \pi\mu C'(\lambda_1\lambda_2)^{-1/2}\ln a + O(a^0) \qquad \text{as } a \to \infty. \quad [5.14]$$

Hence, upon the substitution of the value of C' given by [5.10], the value of $(F_C)_3$ may be written as

$$(F_C)_3 = \mu(\ln a)\frac{3\pi}{2(\lambda_1\lambda_2)^{1/2}}(\Omega_3' - \Omega_3)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{(\lambda_1 - \lambda_2)}{\lambda_1\lambda_2(\lambda_1 + \lambda_2)}\sin\chi\cos\chi + O(a^0). \quad [5.15]$$

6. RESISTANCE MATRICES FOR SURFACES

The results for the force and torque on the surface W as calculated in Sections 2–5 will now be combined into a single relation. Since the forces and torques were calculated on the basis of the creeping motion equations, the relation between these forces and torques and the velocities and angular velocities of W and W' must be linear. Hence if we define a six dimensional force vector $\mathscr{F}$ acting on W as

$$\mathscr{F} = (F_1, F_2, F_3, G_1, G_2, G_3)^T \tag{6.1}$$

and six-dimensional velocity vectors $\mathbf{v}$ and $\mathbf{v}'$ for the surfaces W and W' as

$$\mathbf{v} = (U_1, U_2, U_3, \Omega_1, \Omega_2, \Omega_3)^T, \qquad \mathbf{v}' = (U'_1, U'_2, U'_3, \Omega'_1, \Omega'_2, \Omega'_3)^T \tag{6.2}$$

then there exists (6 × 6) matrices $\mathscr{K}$ and $\mathscr{K}'$ such that $\mathscr{F}$ is given by

$$\mathscr{F} = -\mu(\mathscr{K}\cdot\mathbf{v} + \mathscr{K}'\cdot\mathbf{v}'). \tag{6.3}$$

This type of six dimensional representation was used by Brenner (1964, 1966).

An examination of the results [3.53], [3.54], [4.22], [4.23] and [5.14] shows that the velocities $\mathbf{U}$ and $\mathbf{U}'$ and angular velocities $\mathbf{\Omega}$ and $\mathbf{\Omega}'$ of the surfaces W and W' occur only as the quantities $(\mathbf{U}' - \mathbf{U})$ and $(\mathbf{\Omega}' - \mathbf{\Omega})$. Hence to the order considered

$$\mathscr{K}' = -\mathscr{K} \tag{6.4}$$

so that

$$\mathscr{F} = -\mu\mathscr{K}\cdot(\mathbf{v} - \mathbf{v}'). \tag{6.5a}$$

However it must be noted that [6.4] applies only to the order considered and does not apply, for example, to terms of order (a^0) in the asymptotic expansion for small a. Thus for $a \to 0$

$$\mathscr{F} = -\mu\mathscr{K}\cdot(\mathbf{v} - \mathbf{v}') + O(a^0) \tag{6.5b}$$

where $\mathscr{K}$ tends to infinity as $a \to 0$.

If $\mathbf{v}' = 0$, then $\mathscr{K}$ is the singular part of the (6 × 6) resistance matrix for the surface W in the presence of W'. It was shown by Brenner (1964) that the resistance matrix for a body is symmetric, therefore

$$\mathscr{K} = \mathscr{K}^T \tag{6.6}$$

where $\mathscr{K}^T$ is the transpose of $\mathscr{K}$.

The 21 independent components of $\mathscr{K}$ may be found from [3.53], [3.54], [4.22], [4.23], [5.11], [5.12] and [5.15]. These results may be simplified considerably by using the relations

$$\begin{aligned}
&\left(\frac{1}{2R_1} + \frac{\cos^2\phi}{2S_1} + \frac{\sin^2\phi}{2S_2}\right)\cos\chi - \frac{\sin\phi\cos\phi}{2}\left(\frac{1}{S_1} - \frac{1}{S_1}\right)\sin\chi = \lambda_1\cos\chi \\
&\quad -\left(\frac{1}{2R_2} + \frac{\sin^2\phi}{2S_1} + \frac{\cos^2\phi}{2S_2}\right)\sin\chi + \frac{\sin\phi\cos\phi}{2}\left(\frac{1}{S_1} - \frac{1}{S_2}\right)\cos\chi = -\lambda_1\sin\chi \\
&\quad +\left(\frac{1}{2R_1} + \frac{\cos^2\phi}{2S_1} + \frac{\sin^2\phi}{2S_2}\right)\sin\chi + \frac{\sin\phi\cos\phi}{2}\left(\frac{1}{S_1} - \frac{1}{S_2}\right)\cos\chi = \lambda_2\sin\chi \\
&\quad +\left(\frac{1}{2R_2} + \frac{\sin^2\phi}{2S_1} + \frac{\cos^2\phi}{2S_2}\right)\cos\chi + \frac{\sin\phi\cos\phi}{2}\left(\frac{1}{S_1} - \frac{1}{S_2}\right)\sin\chi = \lambda_2\cos\chi,
\end{aligned} \tag{6.7}$$

and

$$\lambda_1 + \lambda_1 = 1/2R_1 + 1/2R_2 + 1/2S_1 + 1/2S_2, \tag{6.8}$$

which have been derived from [3.12], [3.13], [3.14] and [3.45].

Thus the components of $\mathscr{K}$ are given by

$$\mathscr{K}_{11} = -(\ln a)\frac{\pi}{(\lambda_1\lambda_2)^{1/2}R_1^2}\left[\frac{3\cos^2\chi(1-R_1\lambda_1)^2}{(3\lambda_1+2\lambda_2)\lambda_1}+\frac{3\sin^2\chi(1-R_1\lambda_2)^2}{(2\lambda_1+3\lambda_2)\lambda_2}+R_1^2\right]$$

$$\mathscr{K}_{22} = -(\ln a)\frac{\pi}{(\lambda_1\lambda_2)^{1/2}R_2^2}\left[\frac{3\sin^2\chi(1-R_2\lambda_1)^2}{(3\lambda_1+2\lambda_2)\lambda_1}+\frac{3\cos^2\chi(1-R_2\lambda_2)^2}{(2\lambda_1+3\lambda_2)\lambda_2}+R_2^2\right]$$

$$\mathscr{K}_{33} = (a^{-1})\frac{3\pi}{(\lambda_1\lambda_2)^{1/2}(\lambda_1+\lambda_2)}+O(a^{-1/2})$$

$$\mathscr{K}_{12} = \mathscr{K}_{21} = +(\ln a)\frac{3\pi\sin\chi\cos\chi}{(\lambda_1\lambda_2)^{1/2}R_1R_2}\left[\frac{(1-R_1\lambda_1)(1-R_2\lambda_1)}{(3\lambda_1+2\lambda_2)\lambda_1}-\frac{(1-R_1\lambda_2)(1-R_2\lambda_2)}{(2\lambda_1+3\lambda_2)\lambda_2}\right]$$

$$\mathscr{K}_{13} = \mathscr{K}_{31} = O(\ln a) \qquad \mathscr{K}_{23} = \mathscr{K}_{32} = O(\ln a)$$

$$\mathscr{K}_{44} = -(\ln a)\frac{3\pi}{(\lambda_1\lambda_2)^{1/2}}\left[\frac{\sin^2\chi}{(3\lambda_1+2\lambda_2)\lambda_1}+\frac{\cos^2\chi}{(2\lambda_1+3\lambda_2)\lambda_2}\right]$$

$$\mathscr{K}_{55} = -(\ln a)\frac{3\pi}{(\lambda_1\lambda_2)^{1/2}}\left[\frac{\cos^2\chi}{(3\lambda_1+2\lambda_2)\lambda_1}+\frac{\sin^2\chi}{(2\lambda_1+3\lambda_2)\lambda_2}\right]$$

$$\mathscr{K}_{66} = O(a^0)$$

$$\mathscr{K}_{45} = \mathscr{K}_{54} = +(\ln a)\frac{3\pi\sin\chi\cos\chi}{(\lambda_1\lambda_2)^{1/2}}\left[-\frac{1}{(3\lambda_1+2\lambda_2)\lambda_1}+\frac{1}{(2\lambda_1+3\lambda_2)\lambda_2}\right]$$

$$\mathscr{K}_{46} = \mathscr{K}_{64} = O(a^0), \qquad \mathscr{K}_{56} = \mathscr{K}_{65} = O(a^0)$$

$$\mathscr{K}_{14} = \mathscr{K}_{41} = +(\ln a)\frac{3\pi\sin\chi\cos\chi}{(\lambda_1\lambda_2)^{1/2}R_1}\left[\frac{(1-R_1\lambda_1)}{(3\lambda_1+2\lambda_2)\lambda_1}-\frac{(1-R_1\lambda_2)}{(2\lambda_1+3\lambda_2)\lambda_2}\right]$$

$$\mathscr{K}_{25} = \mathscr{K}_{52} = +(\ln a)\frac{3\pi\sin\chi\cos\chi}{(\lambda_1\lambda_2)^{1/2}R_2}\left[-\frac{(1-R_2\lambda_1)}{(3\lambda_1+2\lambda_2)\lambda_1}+\frac{(1-R_2\lambda_2)}{(2\lambda_1+3\lambda_2)\lambda_2}\right]$$

$$\mathscr{K}_{36} = \mathscr{K}_{63} = +(\ln a)\frac{3\pi\sin\chi\cos\chi}{2(\lambda_1\lambda_2)^{3/2}}\left(\frac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2}\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

$$\mathscr{K}_{15} = \mathscr{K}_{51} = +(\ln a)\frac{3\pi}{(\lambda_1\lambda_2)^{1/2}R_1}\left[\frac{\cos^2\chi(1-R_1\lambda_1)}{(3\lambda_1+2\lambda_2)\lambda_1}+\frac{\sin^2\chi(1-R_1\lambda_2)}{(2\lambda_1+3\lambda_2)\lambda_2}\right]$$

$$\mathscr{K}_{24} = \mathscr{K}_{42} = -(\ln a)\frac{3\pi}{(\lambda_1\lambda_2)^{1/2}R_2}\left[\frac{\sin^2\chi(1-R_2\lambda_1)}{(3\lambda_1+2\lambda_2)\lambda_1}+\frac{\cos^2\chi(1-R_2\lambda_2)}{(2\lambda_1+3\lambda_2)\lambda_2}\right]$$

$$\mathscr{K}_{16} = \mathscr{K}_{61} = O(a^0), \qquad \mathscr{K}_{26} = \mathscr{K}_{62} = O(a^0),$$

$$\mathscr{K}_{34} = \mathscr{K}_{43} = O(\ln a), \qquad \mathscr{K}_{35} = \mathscr{K}_{53} = O(\ln a). \qquad [6.9]$$

Thus of the 21 independent components of $\mathscr{K}_{ij}$, 16 are singular as $a \to 0$ and of these 12 have been explicitly calculated while the other 4 (namely $\mathscr{K}_{13}$, $\mathscr{K}_{23}$, $\mathscr{K}_{43}$ and $\mathscr{K}_{53}$) would require the knowledge of the terms of order r^3 in the forms of the two surfaces given by [2.2a] and [2.3a]. The remaining 5 components (namely $\mathscr{K}_{16}$, $\mathscr{K}_{26}$, $\mathscr{K}_{46}$, $\mathscr{K}_{56}$ and $\mathscr{K}_{66}$) are never singular as $a \to 0$ and their calculation even to the lowest order in a would require the complete outer expansion and would thus require knowledge of the entire particle shapes. However for locally symmetric surfaces W and W' of the forms [2.2b] and [2.3b] the values of $\mathscr{K}$ are as shown above except that:

(i) $\mathscr{K}_{33}$ as shown has an error of order ($\ln a$) instead of order ($a^{-1/2}$); and
(ii) $\mathscr{K}_{13} = \mathscr{K}_{31}$, $\mathscr{K}_{23} = \mathscr{K}_{32}$, $\mathscr{K}_{43} = \mathscr{K}_{34}$ and $\mathscr{K}_{53} = \mathscr{K}_{35}$ are no longer singular, being of order a^0.

In order to find the force and torque about 0′ acting on the surface W' one merely has to interchange the roles of W and W' in the preceding theory.

7. TWO SPHERES ALMOST IN CONTACT

As an example of the use of the results given in the previous section, we consider the surfaces W and W' being respectively that of spheres of radii R and S (see figure 3). It is assumed that the sphere W' is at rest and that W is undergoing a given motion. Hence

$$R_1 = R_2 = R, \qquad S_1 = S_2 = S \tag{7.1}$$

and

$$\mathbf{v}' = 0. \tag{7.2}$$

The six dimensional force vector $\mathscr{F}$ is then given by [6.3] as

$$\mathscr{F} = -\mu \mathscr{K} \cdot \mathbf{v} + O(a^0). \tag{7.3}$$

The angle ϕ may be taken to be zero, the matrix $\mathbf{K}$ given by [3.12], then being

$$\mathbf{K} = \begin{pmatrix} 1/2R + 1/2S & 0 \\ 0 & 1/2R + 1/2S \end{pmatrix}. \tag{7.4}$$

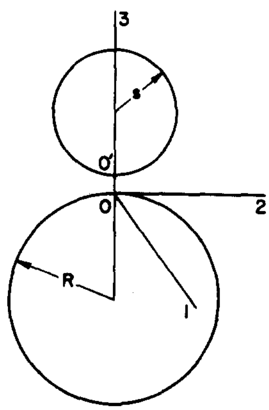

Figure 3. Two spheres almost in contact.

The eigenvalues of **K** are then

$$\lambda_1 = \lambda_2 = 1/2R + 1/2S, \qquad [7.5]$$

the corresponding eigenvectors being indeterminate. Thus χ is also indeterminate and may be taken to be zero. The components of $\mathscr{K}$ given by [6.9] may then be written as

$$\mathscr{K}_{11} = \mathscr{K}_{22} = -(\ln a)\frac{2\pi RS}{(R+S)}\left[\frac{3(S-R)^2}{5(S+R)^2} + 1\right], \qquad \mathscr{K}_{33} = +(a^{-1})\frac{6\pi R^2S^2}{(R+S)^2} + O(\ln a),$$

$$\mathscr{K}_{44} = \mathscr{K}_{55} = -(\ln a)\frac{24\pi R^3S^3}{5(R+S)^3}, \qquad \mathscr{K}_{15} = \mathscr{K}_{51} = +(\ln a)\frac{12\pi R^2S^2(S-R)}{5(R+S)^3},$$

$$\mathscr{K}_{24} = \mathscr{K}_{42} = -(\ln a)\frac{12\pi R^2S^2(S-R)}{5(R+S)^3}, \qquad [7.6]$$

all other components being non-singular.

For the special case of $S = \infty$, the wall W' becomes a plane surface and the problem reduces to that of the motion of a sphere almost in contact with a plane wall. The above values of the components of $\mathscr{K}$ then become

$$\mathscr{K}_{11} = \mathscr{K}_{22} = -(\ln a)16\pi R/5, \qquad \mathscr{K}_{33} = +a^{-1}6\pi R^2 + O(\ln a),$$

$$\mathscr{K}_{44} = \mathscr{K}_{55} = -(\ln a)24\pi R^3/5, \qquad \mathscr{K}_{15} = \mathscr{K}_{51} = -\mathscr{K}_{24} = -\mathscr{K}_{42} = +(\ln a)12\pi R^2/5. \qquad [7.7]$$

Thus by [7.3], it is seen that the force **F** and torque **G** exerted on the sphere is given by

$$F_1 = \mu(\ln a)R(4\pi/5)(4U_1 - 3R\Omega_2), \qquad F_2 = \mu(\ln a)R(4\pi/5)(4U_2 + 3R\Omega_1),$$

$$F_3 = -\mu a^{-1}R^2 6\pi U_3 + O(\ln a),$$

$$G_1 = \mu(\ln a)R^2(12\pi/5)(U_2 + 2R\Omega_1), \qquad G_2 = \mu(\ln a)R^2(12\pi/5)(-U_1 + 2R\Omega_2),$$

$$G_3 = O(a^0). \qquad [7.8]$$

If these equations are expressed relative to the sphere centre rather than the contact point, they become identical with the results for this particular case as calculated by Goldman, Cox & Brenner (1967).

Considering the special case for which the radii of the spheres W and W' are equal so that $R = S$, the components of $\mathscr{K}$ given by [7.6] become

$$\mathscr{K}_{11} = \mathscr{K}_{22} = -(\ln a)\pi R, \qquad \mathscr{K}_{33} = +a^{-1}(3\pi R^2/2) + O(\ln a),$$

$$\mathscr{K}_{44} = \mathscr{K}_{55} = -(\ln a)3\pi R^3/5, \qquad \mathscr{K}_{15} = \mathscr{K}_{51} = \mathscr{K}_{24} = \mathscr{K}_{42} = 0, \qquad [7.9]$$

so that the force **F** and torque **G** exerted on W are given by

$$F_1 = \mu(\ln a)\pi R U_1, \qquad F_2 = \mu(\ln a)\pi R U_2,$$

$$F_3 = -\mu a^{-1}(3\pi/2)R^2 U_3 + O(\ln a)$$

$$G_1 = \mu(\ln a)(3\pi/5)R^3\Omega_1, \quad G_2 = \mu(\ln a)(3\pi/5)R^3\Omega_2, \quad G_3 = O(a^0). \qquad [7.10]$$

These results, when expressed relative to the centre of sphere W as origin, become identical with those given by Zia, Cox & Mason (1967) in their discussion of the behaviour of a chain of spheres in shear flow. Furthermore if one considers the tangential motions of one sphere near a second sphere at rest, the spheres having unequal radii, then the forces and torques on the spheres derived from [7.6] may be shown to be in agreement with those obtained by O'Neill & Majumdar (1970).

As another illustration of the use of the above theory, the force and torque on a sphere of radius R (with surface W) moving close to a fixed circular cylinder of radius S (with surface W') may be calculated. Taking axes such that the 1-axis is parallel to the cylinder axis with the 3-axis normal to the surfaces at O (so that the situation is the same as shown in figure 3 except the sphere of radius S is replaced by cylinder with axis parallel to 1-axis), the radii of curvature are $R_1 = R_2 = R$, $S_1 = \infty$, $S_2 = S$ with $\phi = 0$. Thus matrix **K** given by [3.12] is

$$\mathbf{K} = \begin{pmatrix} 1/2R & 0 \\ 0 & 1/2R + 1/2S \end{pmatrix}$$

whose eigenvalues are

$$\lambda_1 = 1/2R, \qquad \lambda_2 = 1/2R + 1/2S$$

with corresponding eigenvectors

$$\mathbf{x}_1^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{x}_2^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ giving $\chi = 0$. Since the cylinder W' is at rest, $\mathbf{v}' = \mathbf{0}$ giving the 6 dimensional force $\mathscr{F}$ on the sphere as

$$\mathscr{F} = -\mu\mathscr{K}\cdot\mathbf{v} + O(a^0), \qquad [7.11]$$

where the matrix $\mathscr{K}$, determined by [6.9], is given by

$$\mathscr{K}_{11} = -(\ln a)\frac{4\pi R|S|^{1/2}}{|R+S|^{1/2}}\left(\frac{R+4S}{2R+5S}\right)$$

$$\mathscr{K}_{22} = -(\ln a)\frac{4\pi R|S|^{1/2}}{|R+S|^{3/2}}\left(\frac{3R^2+RS+4S^2}{3R+5S}\right)$$

$$\mathscr{K}_{33} = a^{-1}\frac{12\pi R^2|S|^{3/2}}{|R+S|^{1/2}(R+2S)} + O(\ln a)$$

$$\mathscr{K}_{44} = -(\ln a)\frac{24\pi R^3|S|^{5/2}}{|R+S|^{3/2}(3R+5S)},$$

$$\mathscr{K}_{55} = -(\ln a)\frac{24\pi R^3|S|^{3/2}}{|R+S|^{1/2}(2R+5S)},$$

$$\mathscr{K}_{15} = \mathscr{K}_{51} = (\ln a)\frac{12\pi R^2|S|^{3/2}}{|R+S|^{1/2}(2R+5S)},$$

$$\mathscr{K}_{24} = \mathscr{K}_{42} = \frac{-(\ln a)12\pi R^2|S|^{3/2}(S-R)}{|R+S|^{3/2}(3R+5S)}, \qquad [7.12]$$

all other components being non-singular. These results are valid for the sphere either outside ($S > 0$) or inside ($S < 0$, $R + S < 0$) the cylinder.

8. DISCUSSION OF RESULTS

The "lubrication" theory described in the previous sections gives the singular nature as $a \to 0$ of the force and torque exerted on the surface W (with [2.2b]) which is almost in contact with the surface W' (with [2.3b]) at a point O, the two surfaces having prescribed motion.

Thus relative translational motion of the surfaces in the direction of their normals produces a normal force or order a^{-1} and a normal torque of order $(\ln a)$ on the surface W. Also tangential components of velocities and angular velocities of the walls produce a tangential force and torque on W of order $(\ln a)$. Relative rotation of the surfaces about their normal produces a normal force on the surface W of order $(\ln a)$. Also, in general, tangential components of velocity and angular velocity can produce a normal lift force of order $(\ln a)$ and a normal velocity produce tangential components of force and torque of order $(\ln a)$. Although these components have not been determined here their values, which depend on the gradient of surface curvature, have been shown to be no longer singular if the surfaces are locally symmetric.

The results may be used, for example, in investigating the sedimentation of a particle which is almost in contact with a vertical wall. Thus if one considers a uniform sphere sedimenting as a result of a force $(F, 0, 0)$ acting at its centre (see figure 4), it is seen that since this external force and torque on the body must be balanced by the hydrodynamic force and torque given by [7.8], one must have

$$-F_1 = \mu(\ln a)R(4\pi/5)(4U_1 - 3R\Omega_2), \qquad F_1R = \mu(\ln a)R^2(12\pi/5)(-U_1 + 2R\Omega_2), \qquad [8.1]$$

where U_1 is the velocity of the sphere at the contact point O. Thus

$$U_1 = -3R\Omega_2, \qquad [8.2]$$

and if one denotes the 1-component of the velocity at the sphere centre as U_1^* so that

$$U_1^* = U_1 - R\Omega_2, \qquad [8.3]$$

then

$$U_1^*/(R\Omega_2) = -4, \qquad [8.4]$$

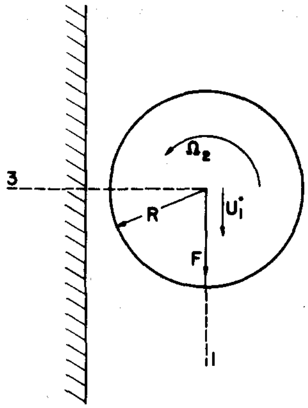

Figure 4. Sphere undergoing sedimentation near a vertical plane wall.

which gives the relationship between the translational velocity and angular rotation of the sphere when the gap a is very small. Also

$$\Omega_2 = \frac{F}{12\pi\mu R^2(\ln a)}, \qquad U_1^* = -\frac{F}{3\pi\mu R(\ln a)} \qquad [8.5]$$

from which it is seen that as $a \to 0$, both Ω_2 and U_1^* tend to zero like $(\ln a)^{-1}$. However as pointed out by Goldman, Cox & Brenner (1967), the theory may no longer be valid if a is too small due to surface roughness effects, Non-Newtonian fluid effects or to the effect of cavitation taking place in the gap. It should be noted that very high pressures are produced in the gap, these being for the general case, of order a^{-2} for relative normal translation, of order $a^{-3/2}$ for tangential relative motion and of order a^{-1} for relative normal rotation of the two surfaces. For the corresponding two dimensional problem of the motion of a circular cylinder almost in contact with a plane, high pressures are similarly produced in the gap (Pinkus & Sternlicht 1961).

The general results given in §7 may in a similar manner be used to investigate the motion of a non-spherical particle undergoing sedimentation near a plane wall or the relative motion of two such particles under the action of external forces (such as electrical forces) in a fluid at rest. The motion of a sphere near a fixed cylinder may be obtained by the use of [7.11] and [7.12] and may form the basis of a model for the motion of a particle in a fibrous filter, or for the motion of a sphere undergoing sedimentation in a cylindrical tube.

In fact if U_1^* is the velocity of centre and Ω_2 the angular velocity of a uniform sphere of radius R undergoing sedimentation inside or outside of a vertical circular cylinder of radius S it may readily be shown from [7.12] that the equations analogous to (8.1) for this case give

$$\frac{U_1^*}{R\Omega_2} = -\left(\frac{4 + R/S}{1 + R/S}\right) \qquad [8.6]$$

where for the sphere outside the cylinder $S > 0$, and for the sphere inside $S < 0$. Note that for this latter case, the value of $U_1^*/R\Omega_2$ increases as R increases and tends to infinity as

$R/S \to -1$, at which stage the theory is no longer valid since one almost has line rather than point contact.

Further applications of the general results given in Section 6 include problems in which the two surfaces involved are:

(a) two cylinders with axes in arbitrary directions,
(b) a plane and an axisymmetric body with fore–aft symmetry (e.g. spheroid, torus) with axis parallel to the plane,
(c) two such axisymmetric bodies with axes parallel,
(d) a circular cylinder and an axisymmetric body (with fore–aft symmetry) with axis perpendicular to cylinder axis.

Also the partial solution to problems involving two completely arbitrary smooth bodies may be found. For example:

(a) If $U_3 \ll U_1, U_2$ (e.g. an arbitrary body undergoing sedimentation near an inclined plane *not* nearly horizontal), the U_1, U_2, Ω_1, Ω_2 can be found if forces and torques are given (leaving only U_3, Ω_3 unknown).
(b) If U_3 is of order U_1, U_2 (e.g. an arbitrary body undergoing sedimentation on a nearly horizontal plane), U_3 may be found.
(c) If body rotates about the 3-axis alone, F_3 may be found.
(d) If body translates along the 3-axis alone, F_3 and G_3 may be found.
(e) In any motion for which $U_3 = 0$ the quantities F_1, F_2, G_1 and G_2 may be found.

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Sommaire—Les forces et les couples agissant sur deux particules solides en mouvement suspendues dans un fluide et presque en contact l'une avec l'autre (ou sur une particule presque en contact avec une paroi) sont trouvées en fonction de leur mouvement relatif en utilisant un type de théorie de lubrification, les résultats ainsi obtenus étant asymptotiquement valables pour des espaces de faible grandeur. Il est supposé que les surfaces des particules impliquées, si elles sont rapprochées, sont telles que le contact aurait lieu à un *seul* point auquel les courbatures de surface sont définies.

Auszug—Die Kräfte und Drehmomente auf zwei sich bewegende feste Trilchen, die in einer Flüssigkeit schweben und sich beinahe berühren (oder auf ein beinahe eine Wand berührendes Teilchen) werden in Bezug auf ihre relative Bewegung mit einer Art Schmiertheorie gefunden, und die so erhaltenen Ergebnisse sind asymptotisch für kleine Lückenbreiten gültig. Es wird angenommen, daß die Oberflächen der betroffenen Teilchen im Falle eines Zusammenkommens derartig sind, daß Berührung an einem *einzelnen* Punkt erfolgen würde, an dem Oberflächenkrümmungen endlich sind.

Резюме—При помощи модели теории смазки нашли с точки зрения относительного движения движущую силу и крутящее усилие двух передвигающихся твердых частиц, взвешенных в жидкости и почти что соприкасающихся друг с другом (или частицы, которая почти что соприкасаестя со стеной). Полученные таким образом результаты оказались асимптотически применимы для нешироких зазоров. Предполагаiот, что если поверхности рассматриваемых частиц столкнутся, тогда контакт произойдет только в той *одной* точке, где кривизны поверхностей финитные.